CONTINUOUS QUANTITATIVE HELLY-TYPE RESULTS

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Abstract. Brazitikos' results on quantititative Helly-type theorems (for the volume and for the diameter) rely on the work of Srivastava on approximate John's decompositions with few vectors. We change this technique by a stronger recent result due to Friedland and Youssef that allow us to obtain Helly-type versions which are sensitive to the number of convex sets involved.

1. Introduction

Helly's classical theorem states that if $\mathcal{C} = \{C_i : i \in I\}$ is a finite family of at least $n+1$ convex sets in $\mathbb{R}^n$ and if any $n+1$ members of $\mathcal{C}$ have non-empty intersection then $\bigcap_{i \in I} C_i$ is non-empty. In general, a Helly-type property is a property Π for which there exists a number $s \in \mathbb{N}$ such that if $\{C_i : i \in I\}$ is a finite family of certain objects and every subfamily of s elements fulfills Π, then the whole family fulfills Π.

In the eighties, Bárány, Katchalski and Pach proved the following quantitative "volume version" of Helly's theorem [BKP82, BKP84]:

Let $\mathcal{C} = \{C_i : i \in I\}$ be a finite family of convex sets in $\mathbb{R}^n$. If the intersection of any $2n$ or fewer members of $\mathcal{H}$ has volume greater than or equal to 1, then $\mathrm{vol}(\bigcap_{i \in I} C_i) \geq c(n)$, where $c(n) > 0$ is a constant depending only on n.

Thus, the previous result express the fact that *"the intersection has large volume"* is a Helly-type property for the family of convex sets.

Since every (closed) convex set is the intersection of a family of closed half-spaces; a simple compactness argument (see [BKP82]) shows that one can remove the restriction that $\mathcal{C}$ is finite and also assume that each convex set is a closed half-space i.e.,

$$\{x \in \mathbb{R}^n : \langle x, v_i \rangle \leq 1\},$$

for some vector $v_i \in \mathbb{R}^n$. Therefore, the theorem of Bárány et. al. is equivalent to the following statement:

Let $\mathcal{H} = \{H_i : i \in I\}$ be a family of closed half-spaces in $\mathbb{R}^n$ such that $\mathrm{vol}(\bigcap_{i \in I} H_i) = 1$. There exist $s \leq 2n$ and $i_1, \dots, i_s \in I$ such that

$$\mathrm{vol}(H_{i_1} \cap \dots \cap H_{i_s})^{1/n} \leq c(n),$$

where $c(n) > 0$ is a constant depending only on n.

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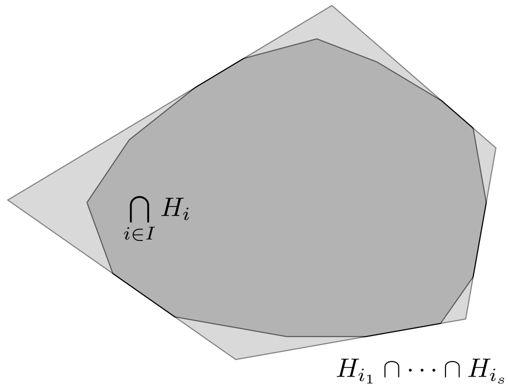

FIGURE 1. A convex body defined as the intersection of half-spaces which is enclosed by a convex set given by the intersection of a few of them.

Of course one cannot replace $2n$ by $2n-1$ in the statement above. Indeed, the cube $[-1/2, 1/2]^n$ in $\mathbb{R}^n$ can be written as the intersection of the $2n$ closed half-spaces

$$H_j^{\pm} := \left\{ x : \langle x, \pm\frac{1}{2}e_j \rangle \leq 1 \right\}$$

and that the intersection of any $2n-1$ of these half-spaces has infinite volume.

The authors of [BKP82] gave the bound $c(n) \leq n^{2n}$ for the constant $c(n)$ and conjectured that one might actually have polynomial growth i.e., $c(n) \leq n^d$ for an absolute constant $d > 0$. Naszódi [Nas16] has verified this conjecture; namely, he proved that $c(n) \leq cn^2$, where $c > 0$ is an absolute constant. A clever but slight refinement of Naszódi's argument, due to Brazitikos [Bra17a, Theorem 3.1.], leads to the exponent $\frac{3}{2}$ instead of 2.

Moreover, Brazitikos showed in [Bra17a, Theorem 1.4.] that if we relax the condition on the number s of half-spaces that we use (but still require that it is proportional to the dimension n) one can improve significantly the estimate, giving a bound of order n.

Theorem 1.1. [Bra17a, Theorem 1.4.] *There exists an absolute constant $\alpha > 0$ with the following property: for every family $\mathcal{H} = \{H_i : i \in I\}$ of closed half-spaces in $\mathbb{R}^n$,*

$$H_i = \{x \in \mathbb{R}^n : \langle x, v_i \rangle \leq 1\},$$

with $\mathrm{vol}(\bigcap H_{i\in I} H_i) = 1$*, there exist $s \leq \alpha n$ and $i_1, \ldots, i_s \in I$ such that*

$$\mathrm{vol}(H_{i_1} \cap \cdots \cap H_{i_s})^{1/n} \leq cn,$$

where $c > 0$ is an absolute constant.

Bárány, Katchalski and Pach also studied the question whether "*the intersection has large diameter*" is a sort of Helly-type property for convex sets. They provided the following quantitative answer to this question:

Let $\{C_i : i \in I\}$ be a family of closed convex sets in $\mathbb{R}^n$ such that $\mathrm{diam}\left(\bigcap_{i\in I} C_i\right) = 1$*. There exist $s \leqslant 2n$ and $i_1, \dots, i_s \in I$ such that*

$$\mathrm{diam}\,(C_{i_1} \cap \cdots \cap C_{i_s}) \leqslant (cn)^{n/2},$$

where $c > 0$ is an absolute constant.

In the same work the authors conjectured that the bound $(cn)^{n/2}$ should be polynomial in n. Leaving aside the requirement that $s \leqslant 2n$, Brazitikos in [Bra17b] provided the following relaxed positive answer:

Theorem 1.2. *There exists an absolute constant $\alpha > 1$ with the following property: if $\{C_i : i \in I\}$ is a finite family of convex bodies in $\mathbb{R}^n$ with* $\mathrm{diam}\left(\bigcap_{i\in I} C_i\right) = 1$*, there exist $s \leqslant \alpha n$ and $i_1, \dots i_s \in I$ such that*

$$\mathrm{diam}(C_{i_1} \cap \cdots \cap C_{i_s}) \leqslant cn^{3/2},$$

where $c > 0$ is an absolute constant.

It should be mentioned that when symmetry is assumed better bounds in both problems can be obtained.

Brazitikos' proofs of Theorem 1.1 and Theorem 1.2 rely on the work of Batson, Spielman and Srivastava on approximate John's decompositions with few vectors [BSS12]. For Theorem 1.1, this is successfully combined with a new and very useful estimate for corresponding 'approximate' Brascamp-Lieb-type inequality while, for Theorem 1.2, the argument is based on a clever lemma of Barvinok from [Bar14]. This lemma in turn, exploits again the theorem of Batson et. al. or to be precise, a more delicate version of Srivastava from [Sri12].

Of course if one is willing to further relax the number of convex sets involved in the statements of Theorems 1.1 and 1.2, then one should expect to obtain better bounds/estimates. The aim of this note is to present the following *continuous* quantitative Helly-type results (i.e., Helly-type results which are sensitive to the number of sets considered).

Theorem 1.3. *(Continuous Helly-type theorem for the volume) Let $1 \leq \delta \leq 2$, there is an absolute constant $\alpha > 1$ with the following property: for every $n \in \mathbb{N}$ and every family $\mathcal{H} = \{H_i : i \in I\}$ of closed half-spaces in $\mathbb{R}^n$,*

$$H_i = \{x \in \mathbb{R}^n : \langle x, v_i \rangle \leq 1\},$$

with $\mathrm{vol}(\bigcap_{i\in I} H_i) = 1$*, there exists $s \leq \alpha n^\delta$ and $i_1, \dots, i_s \in I$ such that*

$$\mathrm{vol}(H_{i_1} \cap \cdots \cap H_{i_s})^{1/n} \leq d_n n^{\frac{3}{2} - \frac{\delta}{2}},$$

where $d_n \to 1$ as $n \to \infty$.

Theorem 1.4. *(Continuous Helly-type theorem for the diameter) Let $1 \leq \delta \leq 2$, there is an absolute constant $\alpha > 1$ with the following property: for every $n \in \mathbb{N}$ and every finite family $\{C_i : i \in I\}$ of convex bodies in $\mathbb{R}^n$ with* $\mathrm{diam}\left(\bigcap_{i\in I} C_i\right) = 1$*, there exist $s \leqslant \alpha n^\delta$ and $i_1, \dots i_s \in I$ such that*

$$\mathrm{diam}(C_{i_1} \cap \cdots \cap C_{i_s}) \leqslant cn^{3 - \frac{3}{2}\delta},$$

where $c > 0$ are absolute constant.

Note that in both theorems we recover the previous mentioned results when the number of sets is linear in n (i.e., when $\delta = 1$). If the number of sets is n^2 then the bounds are the known ones which, of course, follow by directly applying John's classical theorem. Therefore, the dependencies in the exponent of both results obtained seem to be accurate. Moreover, for a linear number of spaces (i.e., $\delta = 1$) the constant that appears in Theorem 1.3 is better than the one in [Bra17a, Theorem 1.4.], since $d_n \to 1$ as n goes to infinity.

To obtain Theorems 1.3 and 1.4 we carefully follow Brazitikos's proofs of Theorems 1.1 and 1.2 but instead of using Batson et. al. or Srivastava's statment on the approximate John's decomposition we replace it with the following stronger result due to Friedland and Youssef (who exploited the recent solution of the Kadison-Singer problem [MSS15], by showing that any $n \times m$ matrix A can be approximated in operator norm by a submatrix with a number of columns of order the stable rank of A).

Theorem 1.5. [FY19, Theorem 4.1] *Let $\{u_j, a_j\}_{1 \leq j \leq m}$ be a John's decomposition of the identity i.e, the identity operator I_n is decomposed in the form $I_n = \sum_{j=1}^m a_j u_j \otimes u_j$. Then for any $\varepsilon > 0$ there exists a multi-set $\sigma \subset [m]$ (i.e., it allows repetitions of the elements) with $|\sigma| \leq n/c\varepsilon^2$ so that*

$$(1-\varepsilon) I_n \preceq \frac{n}{|\sigma|} \sum_{j \in \sigma} (u_j - u) \otimes (u_j - u) \preceq (1+\varepsilon) I_n$$

where $u = \frac{1}{|\sigma|} \sum_{i \in \sigma} u_j$ satisfies $\|u\|_2 \leq \frac{2\varepsilon}{3\sqrt{n}}$, and $c > 0$ is an absolute constant.

In the words of Friedland and Youssef, Thorem 1.5 improves Srivastava's theorem [Sri12, Theorem 5] in three different ways. First the approximation ratio $(1+\varepsilon)/(1-\varepsilon)$ can be made arbitrary close to 1 (while in Srivastava's result one could only get a $(4+\varepsilon)$-approximation). Secondly, it gives an explicit expression of the weights appearing in the approximation. Finally, there is a big difference in the dependence on ε in the estimate of the norm of u: Srivastava obtains a similar bound but with ε replaced by $\sqrt{\varepsilon}$. This behaviour on ε will be crucial for our purposes allowing us to obtain the bounds on our main results. With this at hand, we take the ε parameter small but *depending explicitly* on n.

2. Notation and background

We refer to the book of Artstein-Avidan, Giannopoulos and V. Milman [AAGM15] for basic facts from convexity and asymptotic geometry.

Recall that a convex body in $\mathbb{R}^n$ is a compact convex subset K of $\mathbb{R}^n$ with non-empty interior. We say that the body K is symmetric if $x \in K$ implies that $-x \in K$. For any set X we write $\mathrm{conv}(X)$ for its convex hull. For convex body K we write p_K for the Minkowski's functional of K, that is

$$p_K(x) := \inf\{\lambda > 0 \ : \ x \in \lambda K\}.$$

If $0 \in \mathrm{int}(K)$ then the polar body K° of K is given by

$$K^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leqslant 1 \text{ for all } x \in K\}.$$

Volume is denoted by $\mathrm{vol}(\cdot)$ and diameter by $\mathrm{diam}(\cdot)$. We consider in $\mathbb{R}^n$ the Euclidean structure $\langle \cdot, \cdot \rangle$ and denote by $\|\cdot\|_2$ the corresponding Euclidean norm. We write B_2^n and S^{n-1} for the corresponding Euclidean unit ball and unit sphere respectively.

We say that a convex body K is in John's position if the ellipsoid of maximal volume inscribed in K is the Euclidean unit ball B_2^n. John's classical theorem states that K is in John's position if and only if $B_2^n \subseteq K$ and there exist $u_1, \ldots, u_m \in \mathrm{bd}(K) \cap S^{n-1}$ (contact points of K and B_2^n) and positive real numbers $a_1, \ldots, a_m$ such that

$$\sum_{j=1}^{m} a_j u_j = 0$$

and the identity operator I_n is decomposed in the form

$$I_n = \sum_{j=1}^{m} a_j u_j \otimes u_j, \tag{1}$$

where the rank-one operator $u_j \otimes u_j$ is simply $(u_j \otimes u_j)(y) = \langle u_j, y \rangle u_j$.

If $u_1, \ldots, u_m$ are unit vectors that satisfy John's decomposition (1) with some positive weights a_j. Then, one has the useful equalities

$$\sum_{j=1}^{m} a_j = \mathrm{tr}(I_n) = n \quad \text{and} \quad \sum_{j=1}^{m} a_j \langle u_j, z \rangle^2 = 1$$

for all $z \in S^{n-1}$. Moreover,

$$\mathrm{conv}\{v_1, \ldots, v_m\} \supseteq \frac{1}{n} B_2^n. \tag{2}$$

The body K is in Löwner position if the minimal volume ellipsoid that contains it is the Euclidean ball B_2^n. In that case, we also have a decomposition of the identity as before.

Given two matrices $A, B \in R^{n \times n}$ we write $A \preceq B$ whenever $B - A$ is positive semidefinte.

The letters c, c', C, C' etc. will always denote absolute positive constants which may change from line to line.

3. Continuous Helly-type result for the volume: Theorem 1.3.

As mentioned above we follow the proof of [Bra17a, Theorem 1.4.]. We include all the steps for completeness.

The following Brascamp-Lieb type inequality for approximate John's decomposition of the identity will be crucial.

Theorem 3.1. [Bra17a, Theorem 5.4] *Let $\gamma > 1$. Let $u_1, \cdots, u_s \in S^{n-1}$ and $a_1, \cdots, a_s > 0$ satisfy*

$$Id_n \preceq A := \sum_{j=1}^{s} a_j u_j \otimes u_j \preceq \gamma Id_n$$

and let $k_j = a_j\langle A^{-1}u_j, u_j\rangle > 0,\ 1 \le j \le s$. *If* $f_1, \cdots, f_s : \mathbb{R} \longrightarrow \mathbb{R}^+$ *integrable functions then*

$$\int\limits_{\mathbb{R}^n} \prod_{j=1}^{s} f_j^{k_j}\left(\langle x, u_j\rangle\right) \mathrm{d}x \le \gamma^{\frac{n}{2}} \prod_{j=1}^{s} \left(\int\limits_{\mathbb{R}} f_j(t)\,\mathrm{d}t\right)^{k_j}.$$

We now prove Theorem 1.3.

Proof. (of Theorem 1.3)

Without loss of generality we assume that $P := \bigcap_{i\in I} H_i$ is in John's position. Therefore there exist $J \subseteq I$ and vectors $(u_j)_{j\in J}$ which are contact points between P and S^{n-1} and $(a_j)_{j\in J}$ positive numbers, such that

$$Id_n = \sum_{j\in J} a_j u_j \otimes u_j \quad \text{and} \quad \sum_{j\in J} a_j u_j = 0.$$

Using Friedland and Youssef's approximate decomposition, Theorem 1.5, we can find a multi-set $\sigma \subseteq J$ with $|\sigma| \le \frac{n}{c\varepsilon^2}$ and a vector $u = \frac{-1}{|\sigma|}\sum_{j\in\sigma} u_j$ such that

$$(1-\varepsilon)Id_n \preceq \frac{n}{|\sigma|}\sum_{j\in\sigma}(u_j+u)\otimes(u_j+u) \preceq (1+\varepsilon)Id_n,$$

also satisfying that $\frac{n}{|\sigma|}\sum_{j\in\sigma} u_j + u = 0$ and $|u| \le \frac{2\varepsilon}{3\sqrt{n}}$.

We consider the vector $w := \frac{3u}{2\sqrt{n}\varepsilon}$. Recall that $\frac{1}{n}B_2^n \subseteq \mathrm{conv}\{u_j,\ j\in J\}$, thus $\|w\|_2 \le \frac{1}{n}$ and hence $w \in \mathrm{conv}\{u_j,\ j\in J\}$. By Carathéodory's Theorem, we know that there is $\tau \subseteq J$, with $|\tau| \le n+1$ and $\rho_i > 0,\ i\in\tau$ such that

$$w = \sum_{i\in\tau}\rho_i u_i \quad \text{and} \quad \sum_{i\in\tau}\rho_i = 1.$$

Also notice that, since $u = \frac{-1}{|\sigma|}\sum_{j\in\sigma} u_j$ and $\sum_{j\in\sigma}\frac{1}{|\sigma|} = 1$, $-u \in \mathrm{conv}\{u_j,\ j\in\sigma\}$. Therefore, we have that the segment $\left[-u, \frac{3u}{2\sqrt{n}\varepsilon}\right]$ is contained in $\mathrm{conv}\{u_j,\ j\in\sigma\cup\tau\}$. For $j\in\sigma$ we define

$$v_j := \sqrt{\frac{n}{n+1}}\left(-u_j, \frac{1}{\sqrt{n}}\right) \quad \text{and} \quad b_j = \frac{n+1}{|\sigma|}.$$

Set $v := -\sqrt{\frac{n}{n+1}}\,(u,0)$. So, we have

$$\begin{aligned}
\sum_{j\in\sigma} b_j(v_j+v)\otimes(v_j+v) &= \sum_{j\in\sigma}\frac{n}{|\sigma|}\left(-(u_j+u), \frac{1}{\sqrt{n}}\right)\otimes\left(-(u_j+u), \frac{1}{\sqrt{n}}\right)\\
&= \begin{pmatrix} \sum_{j\in\sigma}\frac{n}{|\sigma|}(u_j+u)\otimes(u_j+u) & \frac{\sqrt{n}}{|\sigma|}\sum_{j\in\sigma}(u_j+u)\\ \frac{\sqrt{n}}{|\sigma|}\sum_{j\in\sigma}(u_j+u)^t & \frac{n}{|\sigma|}\sum_{j\in\sigma}\frac{1}{n}\end{pmatrix}\\
&= \begin{pmatrix} \sum_{j\in\sigma}\frac{n}{|\sigma|}(u_j+u)\otimes(u_j+u) & 0\\ 0 & 1\end{pmatrix},
\end{aligned}$$

which implies

$$(1-\varepsilon)Id_{n+1} \preceq \sum_{j\in\sigma} b_j(v_j+v)\otimes(v_j+v) \preceq (1+\varepsilon)Id_{n+1}. \tag{3}$$

The sum $\sum_{j\in\sigma} b_j(v_j+v)\otimes(v_j+v)$ can be written as

$$\sum_{j\in\sigma} b_j v_j\otimes v_j + v\otimes\left(\sum_{j\in\sigma} b_j v_j\right) + \left(\sum_{j\in\sigma} b_j v_j\right)\otimes v + (n+1)v\otimes v,$$

and notice that since

$$\begin{aligned}\sum_{j\in\sigma} b_j v_j &= \sum_{j\in\sigma}\frac{n+1}{|\sigma|}\sqrt{\frac{n}{n+1}}\left(-u_j,\frac{1}{\sqrt{n}}\right)\\ &= \sqrt{\frac{n+1}{n}}\left(-\sum_{j\in\sigma}\frac{n}{|\sigma|}u_j, \frac{1}{|\sigma|}\sum_{j\in\sigma}\sqrt{n}\right)\\ &= \sqrt{\frac{n+1}{n}}\left(nu,\sqrt{n}\right),\end{aligned}$$

we obtain that

$$\left(\sum_{j\in\sigma} b_j v_j\right)\otimes v = \sqrt{\frac{n+1}{n}}\left(nu,\sqrt{n}\right)\otimes\sqrt{\frac{n}{n+1}}(-u,0) = \begin{pmatrix}-nu\otimes u & 0\\ -\sqrt{n}u^t & 0\end{pmatrix},$$

$$v\otimes\left(\sum_{j\in\sigma} b_j v_j\right) = \begin{pmatrix}-nu\otimes u & -\sqrt{n}u\\ 0 & 0\end{pmatrix},$$

and $(n+1)v\otimes v = \begin{pmatrix}nu\otimes u & 0\\ 0 & 0\end{pmatrix}$.

Hence, we can write Equation (3) as

$$(1-\varepsilon)Id_{n+1} - T \preceq \sum_{j\in\sigma} b_j v_j\otimes v_j \preceq (1+\varepsilon)Id_{n+1},$$

where $T = v\otimes\left(\sum_{j\in\sigma} b_j v_j\right) + \left(\sum_{j\in\sigma} b_j v_j\right)\otimes v + (n+1)v\otimes v = \begin{pmatrix}V & z\\ z & 0\end{pmatrix}$, with $V = -nu\otimes u$ y $z = -\sqrt{n}u$. Now, for $(x,t)\in S^n$ we have that

$$\begin{aligned}\langle T(x,t),(x,t)\rangle &= \langle (Vx+zt,\langle z,x\rangle),(x,t)\rangle\\ &= \langle (Vx,0),(x,t)\rangle + \langle (zt,\langle z,x\rangle),(x,t)\rangle\\ &\le \langle Vx,x\rangle + |(zt,\langle z,x\rangle)||(t,x)| = \langle Vx,x\rangle + \left(|zt|^2+\langle z,x\rangle^2\right)^{\frac{1}{2}}\\ &\le \|V\||x|^2 + \left(|z|^2t^2+|z|^2|x|^2\right)^{\frac{1}{2}} \le \|V\| + |z|\left(t^2+|x|^2\right)^{\frac{1}{2}}\\ &= \|V\| + |z||(x,t)| = \|V\| + |z| = n|u|^2 + \sqrt{n}|u|\\ &\le n\frac{4\varepsilon^2}{9n} + \sqrt{n}\frac{2\varepsilon}{3\sqrt{n}} = \frac{4\varepsilon^2}{9} + \frac{2\varepsilon}{3}\\ &\le \varepsilon,\end{aligned}$$

for ε small enough (say $\varepsilon\le\frac{3}{4}$). So, $\|T\|\le\varepsilon$, and hence Equation (3) implies that

$$(1-2\varepsilon)Id_{n+1} \preceq A := \sum_{j\in\sigma} b_j v_j\otimes v_j \preceq (1+2\varepsilon)Id_{n+1},$$

or equivalently

$$Id_{n+1} \preceq \sum_{j\in\sigma} \frac{b_j}{1-2\varepsilon} v_j \otimes v_j \preceq \gamma Id_{n+1},$$

with $\gamma = \frac{1+2\varepsilon}{1-2\varepsilon}$. Applying Theorem 3.1, if $f_j : \mathbb{R} \to \mathbb{R}^+$ are measurable functions, then

$$\int\limits_{\mathbb{R}^{n+1}} \prod_{j\in\sigma} f_j^{k_j}\left(\langle x, v_j\rangle\right) \mathrm{d}x \leq \gamma^{\frac{n+1}{2}} \prod_{j\in\sigma} \left(\int\limits_{\mathbb{R}} f_j(t)\,\mathrm{d}t\right)^{k_j},$$

where

$$k_j = \frac{b_j}{1-2\varepsilon}\left\langle \left(\frac{1}{1-2\varepsilon}A\right)^{-1} v_j, v_j \right\rangle = b_j \langle A^{-1} v_j, v_j\rangle.$$

Since $A^{-1} \preceq \frac{1}{1-2\varepsilon} Id_{n+1}$, we have that $\frac{k_j}{b_j} \leq \frac{1}{1-2\varepsilon}$. Now for $j \in \sigma$ we consider $f_j(t) := e^{\frac{-b_j}{k_j}t}\chi_{[0,\infty)}(t)$. So,

$$\begin{aligned}
\int\limits_{\mathbb{R}^{n+1}} \prod_{j\in\sigma} f_j^{k_j}\left(\langle x, v_j\rangle\right) \mathrm{d}x &\leq \gamma^{\frac{n+1}{2}} \prod_{j\in\sigma} \left(\int\limits_{\mathbb{R}} f_j(t)\,\mathrm{d}t\right)^{k_j} \\
&= \gamma^{\frac{n+1}{2}} \prod_{j\in\sigma} \frac{k_j}{b_j}^{k_j} \\
&\leq \gamma^{\frac{n+1}{2}} \frac{1}{(1-2\varepsilon)^{\sum_{j\in\sigma} k_j}} = \gamma^{\frac{n+1}{2}} \frac{1}{(1-2\varepsilon)^{n+1}} \\
&= \left(\frac{1+2\varepsilon}{(1-2\varepsilon)^3}\right)^{\frac{n+1}{2}}
\end{aligned}$$

Set

$$Q = \bigcap_{i\in\sigma\cup\tau} H_i = \{x \in \mathbb{R}^n : \langle x, u_j\rangle < 1,\ j \in \sigma\cup\tau\},$$

and let $y = (x, r) \in \mathbb{R}^{n+1}$. Assume that $r > 0$ and $x \in \frac{r}{\sqrt{n}}Q$. Then we have that $\langle x, u_j\rangle < \frac{r}{\sqrt{n}}$ for every $j \in \sigma$, which implies that $\langle y, v_j\rangle > 0$ for every $j \in \sigma$, and then $\prod\limits_{j\in\sigma} f_j^{k_j}(\langle y, v_j\rangle) > 0$. We also have that

$$\begin{aligned}
\left\langle \frac{1}{|\sigma|}\sum_{j\in\sigma} u_j, x\right\rangle &= \langle -u, x\rangle = \frac{2\sqrt{n}\varepsilon}{3}\langle -w, x\rangle \\
&= \frac{2\sqrt{n}\varepsilon}{3}\left\langle -\sum_{i\in\tau}\rho_i u_i, x\right\rangle \geq \frac{-2\sqrt{n}\varepsilon}{3}\left(\sum_{i\in\tau}\rho_i\right)\frac{r}{\sqrt{n}} \\
&= \frac{-2\varepsilon r}{3}.
\end{aligned}$$

Thus, if $y=(x,r)\in \frac{r}{\sqrt{n}}Q\times(0,\infty)$, then

$$\begin{aligned}\prod_{j\in\sigma} f_j^{k_j}(\langle y,v_j\rangle) &= exp\left(-\sum_{j\in\sigma} b_j\left(\frac{r}{\sqrt{n+1}}-\sqrt{\frac{n}{n+1}}\langle x,u_j\rangle\right)\right)\\ &= exp\left(\frac{-r}{\sqrt{n+1}}\sum_{j\in\sigma} b_j\right) exp\left(\sqrt{n}\sqrt{n+1}\left\langle x,\frac{1}{|\sigma|}\sum_{j\in\sigma} u_j\right\rangle\right)\\ &\geq e^{-r\sqrt{n+1}}e^{-\sqrt{n}\sqrt{n+1}\frac{2}{3}r\varepsilon}=e^{-r\sqrt{n+1}\left(1+\frac{2}{3}\varepsilon\sqrt{n}\right)}.\end{aligned}$$

Now, by Theorem 3.1,

$$\begin{aligned}\frac{\mathrm{vol}(Q)}{n^{\frac{n}{2}}}\int_0^\infty r^n e^{-r\sqrt{n+1}\left(1+\frac{2}{3}\varepsilon\sqrt{n}\right)}\,\mathrm{d}r &= \int_0^\infty\int_{\frac{r}{\sqrt{n}}Q} e^{-r\sqrt{n+1}\left(1+\frac{2}{3}\varepsilon\sqrt{n}\right)}\,\mathrm{d}x\,\mathrm{d}r\\ &\leq \int_{\mathbb{R}^{n+1}}\prod_{j\in\sigma} f_j^{k_j}(\langle y,v_j\rangle)\,\mathrm{d}y\\ &\leq \left(\frac{1+2\varepsilon}{(1-2\varepsilon)^3}\right)^{\frac{n+1}{2}}.\end{aligned}$$

Using that $B_2^n\subseteq P$, and the fact that

$$\int_0^\infty r^n e^{-r\sqrt{n+1}\left(1+\frac{2}{3}\varepsilon\sqrt{n}\right)}\,\mathrm{d}r=\frac{n!}{(n+1)^{\frac{n+1}{2}}\left(1+\frac{2}{3}\varepsilon\sqrt{n}\right)^{n+1}},$$

we obtain, by taking $1+\varepsilon'=\frac{1+2\varepsilon}{(1-2\varepsilon)^3}$,

$$\begin{aligned}\mathrm{vol}\Big(\bigcap_{i\in\sigma\cup\tau} H_i\Big)=\mathrm{vol}(Q) &\leq \frac{(1+\varepsilon')^{\frac{n+1}{2}}n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}\left(1+\frac{2}{3}\varepsilon\sqrt{n}\right)^{n+1}}{n!}\frac{\mathrm{vol}(P)}{\mathrm{vol}(B_2^n)}\\ &= \frac{(1+\varepsilon')^{\frac{n+1}{2}}n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}\left(1+\frac{2}{3}\varepsilon\sqrt{n}\right)^{n+1}}{n!}\frac{\Gamma\left(\frac{n}{2}+1\right)\mathrm{vol}(P)}{\pi^{\frac{n}{2}}}.\end{aligned}$$

By Stirling's formula we get, for a constant $C>0$, the inequality

$$\begin{aligned}\mathrm{vol}(\bigcap_{i\in\sigma\cup\tau} H_i) &\leq C\frac{(1+\varepsilon')^{\frac{n+1}{2}}n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}\left(1+\frac{2}{3}\varepsilon\sqrt{n}\right)^{n+1}}{\pi^{\frac{n}{2}}\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}\frac{n}{2}\sqrt{\frac{4\pi}{n}}\left(\frac{n}{2e}\right)^{\frac{n}{2}}\mathrm{vol}(P)\\ &= C(1+\varepsilon')^{\frac{n+1}{2}}\frac{\left(1+\frac{2}{3}\varepsilon\sqrt{n}\right)^{n+1}n^n n(n+1)^{\frac{n+1}{2}}}{n^n n}\left(\frac{e}{2\pi}\right)^{\frac{n}{2}}\frac{1}{\sqrt{2}}\mathrm{vol}(P)\\ &= C(1+\varepsilon')^{\frac{n+1}{2}}\left(1+\frac{2}{3}\varepsilon\sqrt{n}\right)^{n+1}(n+1)^{\frac{n+1}{2}}\left(\frac{e}{2\pi}\right)^{\frac{n}{2}}\frac{1}{\sqrt{2}}\mathrm{vol}(P).\end{aligned}$$

Fix $\varepsilon:=\frac{1}{4}n^{(1-\delta)/2}$, using that $1+\varepsilon'=\frac{1+2\varepsilon}{(1-2\varepsilon)^3}$ we have

$$(1+\varepsilon')\left(1+\frac{2}{3}\varepsilon\sqrt{n}\right)^2\frac{e}{2\pi}=(1+\varepsilon')\left(1+\frac{1}{6}n^{(2-\delta)/2}\right)^2\frac{e}{2\pi}<cn^{2-\delta}.$$

Therefore,

$$\begin{aligned}
\mathrm{vol}(\bigcap_{i\in\sigma\cup\tau} H_i) &\leq Cn^{\frac{n}{2}}\left(1+\frac{1}{n}\right)^{\frac{n}{2}}\sqrt{n+1}\, n^{(2-\delta)/2}\, n^{n(2-\delta)/2}\,\frac{\sqrt{2\pi}}{\sqrt{2e}}\,\mathrm{vol}(P)\\
&\leq C_1\sqrt{\frac{e(n+1)\pi}{e}}\, n^{\frac{n}{2}}\, n^{(2-\delta)/2}\, n^{n(2-\delta)/2}\,\mathrm{vol}(P)\\
&= \left(\underbrace{C_1\sqrt{n+1}n^{(2-\delta)/2}\sqrt{\pi}}_{C_n}\right) n^{n(3-\delta)/2}\,\mathrm{vol}(P).
\end{aligned}$$

We conclude that

$$\mathrm{vol}(\bigcap_{i\in\sigma\cup\tau} H_i) \leq C_n\, n^{n(3-\delta)/2}\ \mathrm{vol}(P),$$

where the intersection is taken over at most $|\sigma\cup\tau| \leq \frac{n}{c\varepsilon^2}+n+1 = \frac{n^\delta}{c}+n+1 \leq \alpha n^\delta$ half-spaces. Since the constant C_n is of order $n^{(3-\delta)/2}$, we have that $d_n := C_n^{1/n} \to 1$ as $n\to\infty$.

It should be mentioned that the case $\delta = 2$ is of course easier (we just use John's decomposition of the identity and the classical Brascamp-Lieb inequality directly). □

4. Continuous Helly-type theorem for the diameter

To obtain Theorem 1.4 we prove the following proposition, which is a continuous version of [Bra17b, Proposition 4.2.]. We feel it is interesting in its own right. Again we include all the steps for completeness.

Proposition 4.1. *Let $1 \leq \delta \leq 2$. If K is a convex body whose minimal volume ellipsoid is the Euclidean unit ball, then there is a subset $X \subseteq K\cap S^{n-1}$ of cardinality $card(X) \leq \alpha n^\delta$ and*

$$K \subseteq B_2^n \subseteq Cn^{2-\frac{\delta}{2}}\,\mathrm{conv}(X),$$

where $\alpha, C > 0$ are absolute constant.

Proof. By John's theorem there exist $v_j \in K\cap S^{n-1}$ and $a_j > 0$, $j\in J$, such that

$$I_n = \sum_{j\in J} a_j v_j\otimes v_j \text{ and } \sum_{j\in J} a_j v_j = 0.$$

Let $\varepsilon > 0$ small enough to be fixed later. By Theorem 1.5 we can find a multiset $\sigma\subseteq J$ of cardinal $|\sigma| \leq \frac{n}{c\varepsilon^2}$ such that

$$(1-\varepsilon)I_n \preceq \frac{n}{\sigma}\sum_{j\in\sigma}(v_j+v)\otimes(v_j+v) \preceq (1+\varepsilon)I_n,$$

where $v = \frac{-1}{|\sigma|}\sum_{j\in\sigma} v_j$ satisfies $\|v\|_2 \leq \frac{2\varepsilon}{3\sqrt{n}}$.

Then, the vector $w = \frac{3v}{2\sqrt{n}\varepsilon}$ satisfies $\|w\|_2 \leq \frac{1}{n}$ and therefore by Equation (2), it belongs to $\mathrm{conv}\{v_j :\ j\in J\}$. By Carathéodory's theorem there exist $\tau\subseteq J$ with $|\tau| \leq n+1$ and $\rho_i > 0$, $i\in\tau$ such that

$$w = \sum_{i\in\tau}\rho_i v_i,\ \text{and } \sum_{i\in\tau}\rho_i = 1.$$

Observe also that $-v = \frac{1}{|\sigma|}\sum_{j\in\sigma} v_j$ is in $\mathrm{conv}\{v_j :\ j\in\sigma\}$. Let

$$T := \frac{n}{|\sigma|}\sum_{j\in\sigma} v_j\otimes v + \frac{n}{|\sigma|}\sum_{j\in\sigma} v\otimes v_j + v\otimes v.$$

As in the proof of Theorem 1.3 it is easy to see that $|\langle Tx, x\rangle| \le \varepsilon$ for every unit vector $x\in\mathbb{R}^n$ (provided that ε is small enough). Thus

$$(1-2\varepsilon)I_n \preceq (1-\varepsilon)I_n - T \preceq \frac{n}{|\sigma|}\sum_{j\in\sigma} v_j\otimes v_j \preceq (1+\varepsilon)I_n - T \preceq (1+2\varepsilon)I_n.$$

Define $X := \{v_j : j\in\sigma\cup\tau\}$ and $E := \mathrm{conv}(X)$. Let us show that $B_2^n \subseteq c\varepsilon n^{3/2}E$. Indeed, let $x\in S^{n-1}$; set $A := \frac{n}{|\sigma|}\sum_{j\in\sigma} v_j\otimes v_j$ and $\rho := \min\{\langle x, v_j\rangle :\ j\in\sigma\}$. Note that $|\rho|\le 1$ and $\langle x, vj\rangle - \rho \le 2$ for all $j\in\sigma$.

If $\rho < 0$ we have

$$\begin{aligned}
p_E(Ax) &\le p_E\left(Ax - \rho\frac{n}{|\sigma|}\sum_{j\in\sigma} v_j\right) + p_E\left(\rho\frac{n}{|\sigma|}\sum_{j\in\sigma} v_j\right)\\
&= p_E\left(\sum_{j\in\sigma}\frac{n}{|\sigma|}(\langle x, v_j\rangle - \rho)v_j\right) + p_E(n\rho(-v))\\
&\le \sum_{j\in\sigma}\frac{n}{|\sigma|}(\langle x, v_j\rangle - \rho)p_E(v_j) - n\rho p_E(v)\\
&\le n\left(2 + \frac{2\sqrt{n}\varepsilon}{3}p_E(w)\right)\\
&\le c_1\varepsilon n^{3/2},
\end{aligned}$$

where we are using that $w\in K$ and therefore $p_E(w)\le 1$.

On the other hand, if $\rho\ge 0$, then $\langle x, v_j\rangle \ge 0$ for all $j\in\sigma$, therefore

$$p_E(Ax) = p_E\left(\frac{n}{|\sigma|}\sum_{j\in\sigma}\langle x, v_j\rangle v_j\right) \le \frac{n}{|\sigma|}\sum_{j\in\sigma}\langle x, v_j\rangle p_E(v_j) \le n.$$

This say that

$$p_{A^{-1}(E)}(x) \le c_2\varepsilon n^{3/2}$$

for all $x\in S^{n-1}$, where $c_2 > 0$ is an absolute constant.

Therefore we have

$$(1-2\varepsilon)B_2^n \subseteq A(B_2^n) \subseteq c_2(1+2\varepsilon)\varepsilon n^{3/2}E.$$

Finally, fix $\varepsilon := \frac{1}{4}n^{\frac{1}{2}-\frac{\delta}{2}}$. Since K is in Löwner's position

$$K\subseteq B_2^n \subseteq C_2\frac{1+2\varepsilon}{1-2\varepsilon}\varepsilon n^{3/2} \subseteq Cn^{2-\frac{\delta}{2}}\,\mathrm{conv}(X),$$

with $|X| = |\sigma\cup\tau| \le cn^\delta + n + 1 \le \alpha n^\delta$. $\square$

Let us now see the proof of the Theorem 1.4.

Proof. Consider $P := \bigcap_{i\in I} C_i$. Without loss of generality we can assume that $0 \in int(P)$ and that the polar body

$$P^\circ = \operatorname{conv}(\bigcap_{i\in I} C_i^\circ)$$

is in Löwner's position. Using Proposition 4.1 for the body $K = P^\circ$, we know there exists a set $X = \{v_1, \cdots, v_s\} \subseteq P^\circ \cap S^{n-1}$ such that $|X| \leq \alpha n^\delta$ and

$$P^\circ \subseteq Cn^{2-\frac{\delta}{2}} \operatorname{conv}(X),$$

where $C > 0$ is an absolute constant. Since $v_1, \cdots, v_s$ are contact points between P° and B_2^n, then we have that $v_j \in \bigcap_{i\in I} C_i^\circ$ for all $j = 1, \cdots, s$. This implies that there exist $s \leq \alpha n^\delta$ and bodies $\{C_{i_j}\}$, such that $v_j \in C_{i_j}^\circ$ for all $j = 1 \cdots, s$. Then $\operatorname{conv}(X) \subseteq \operatorname{conv}(C_{i_1}^\circ \cup \cdots \cup C_{i_s}^\circ)$ and hence

$$P^\circ \subseteq Cn^{2-\frac{\delta}{2}} \operatorname{conv}(C_{i_1}^\circ \cup \cdots \cup C_{i_s}^\circ).$$

This shows that

$$C_{i_1} \cap \cdots \cap C_{i_s} \subseteq cn^{2-\frac{\delta}{2}} P,$$

and therefore we have the following estimate for the diameter

$$\operatorname{diam}(C_{i_1} \cap \cdots \cap C_{i_s}) \leq cn^{2-\frac{\delta}{2}}.$$

This concludes the proof. □

5. Final comments: symmetry assumption

It is well-known that if all the bodies are symmetric the bounds for these kind of results are better (see, for example, [Bra17a, Theorem 1.2] and [Bra17b, Theorem 1.2.]). In that case, for a linear number of convex sets, the bounds are of order $n^{1/2}$. One should be tempted to think that relaxing the number of sets in these statements provides again stronger estimates but, unfortunately, we cannot have these type of continuous versions as before. Indeed, the exponent in n in cannot be improved by allowing more sets: for example we can find $w_1, \ldots, w_N \in S^{n-1}$ (assuming that N is exponential in the dimension n) such that

$$B_2^n \subseteq \bigcap_{j=1}^{N} H_j \subseteq 2B_2^n, \tag{4}$$

where H_j is defined as the strip

$$H_j = \{x \in \mathbb{R}^n : |\langle x, w_j\rangle| \leqslant 1\}.$$

Thus, if $s = n^\delta$ with $\delta > 1$, for any choice of $j_1, \ldots, j_s \in \{1, \ldots, N\}$ we can use the classical lower bound for the volume due to Carl-Pajor [CP88] and Gluskin [Glu89], which shows that

$$|H_{j_1} \cap \cdots \cap H_{j_s}|^{1/n} \geqslant \frac{C}{\sqrt{\log(n)}}. \tag{5}$$

Therefore, if $H_{j_1} \cap \cdots \cap H_{j_s} \subseteq \beta \bigcap_{j=1}^{N} H_j$ for some $\beta > 0$, by comparing its volumes we obtain that

$$\beta \geqslant \frac{|H_{j_1} \cap \cdots \cap H_{j_s}|^{1/n}}{|2B_2^n|^{1/n}} \geqslant c\frac{\sqrt{n}}{\sqrt{\log n}}, \tag{6}$$

where $c > 0$ is an absolute constant.

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References

[AAGM15] Shiri Artstein-Avidan, Apostolos Giannopoulos, and Vitali D Milman. *Asymptotic Geometric Analysis, Part I*, volume 202. American Mathematical Soc., 2015.

[Bar14] Alexander Barvinok. Thrifty approximations of convex bodies by polytopes. *International Mathematics Research Notices*, 2014(16):4341–4356, 2014.

[BKP82] Imre Bárány, Meir Katchalski, and Janos Pach. Quantitative Helly-type theorems. *Proceedings of the American Mathematical Society*, 86(1):109–114, 1982.

[BKP84] Imre Barany, Meir Katchalski, and Janos Pach. Helly's theorem with volumes. *The American Mathematical Monthly*, 91(6):362–365, 1984.

[Bra17a] Silouanos Brazitikos. Brascamp–Lieb inequality and quantitative versions of Helly's theorem. *Mathematika*, 63(1):272–291, 2017.

[Bra17b] Silouanos Brazitikos. Quantitative Helly-type theorem for the diameter of convex sets. *Discrete & Computational Geometry*, 57(2):494–505, 2017.

[BSS12] Joshua Batson, Daniel A Spielman, and Nikhil Srivastava. Twice-Ramanujan sparsifiers. *SIAM Journal on Computing*, 41(6):1704–1721, 2012.

[CP88] Bernd Carl and Alain Pajor. Gelfand numbers of operators with values in a Hilbert space. *Inventiones mathematicae*, 94(3):479–504, 1988.

[FY19] Omer Friedland and Pierre Youssef. Approximating matrices and convex bodies. *International Mathematics Research Notices*, 2019(8):2519–2537, 2019.

[Glu89] Efim Gluskin. Extremal properties of orthogonal parallelepipeds and their applications to the geometry of banach spaces. *Mathematics of the USSR-Sbornik*, 64(1):85, 1989.

[MSS15] Adam W Marcus, Daniel A Spielman, and Nikhil Srivastava. Interlacing families ii: Mixed characteristic polynomials and the Kadison–Singer problem. *Annals of Mathematics*, 182:327–350, 2015.

[Nas16] Márton Naszódi. Proof of a conjecture of Bárány, Katchalski and Pach. *Discrete & Computational Geometry*, 55(1):243–248, 2016.

[Sri12] Nikhil Srivastava. On contact points of convex bodies. In *Geometric aspects of functional analysis*, pages 393–412. Springer, 2012.

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