Geometry of integral polynomials, *M*-ideals and unique norm preserving extensions ☆

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Abstract

We use the Aron–Berner extension to prove that the set of extreme points of the unit ball of the space of integral k-homogeneous polynomials over a real Banach space X is $\{\pm\phi^k \colon \phi \in X^*, \ \|\phi\| = 1\}$. With this description we show that, for real Banach spaces X and Y, if X is a nontrivial M-ideal in Y, then $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X$ (the k-th symmetric tensor product of X endowed with the injective symmetric tensor norm) is *never* an M-ideal in $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} Y$. This result marks up a difference with the behavior of nonsymmetric tensors since, when X is an M-ideal in Y, it is known that $\widehat{\bigotimes}_{\varepsilon_k}^{k} X$ (the k-th tensor product of X endowed with the injective tensor norm) is an M-ideal in $\widehat{\bigotimes}_{\varepsilon_k}^{k} Y$. Nevertheless, if X is also Asplund, we prove that every integral k-homogeneous polynomial in X has a unique extension to Y that preserves the integral norm. Other applications to the metric and isomorphic theory of symmetric tensor products and polynomial ideals are also given.

0. Introduction

The study of extreme points of dual unit balls and the presence of M-ideal structures showed to be very useful tools in the theory of Banach spaces, leading to a better understanding of the geometry of the spaces involved.

Ruess–Stegall [36], Ryan–Turett [37], Boyd–Ryan [9], Dineen [22] and Boyd–Lassalle [8] in their investigations studied the extreme points of the unit ball of the space of (integral) polynomials defined on a Banach space. On the other hand, a number of authors have examined M-ideal structures in tensor products, operator spaces, spaces of polynomials or Banach algebras, see e.g. Werner [40,39], Dimant [20], Lima [32] and Harmand–Werner–Werner [28] (see also the references therein).

Motivated by the increasing interest in the theory of homogeneous polynomials and symmetric tensor products, we study the extreme points of the unit ball of a space of integral polynomials, the existence of an M-ideal structure in the symmetric injective tensor product and unique norm preserving extensions for integral polynomials (and also for polynomials belonging to other ideals).

In 1972 Alfsen and Effros [2] introduced the notion of an M-ideal in a Banach space. The presence of an M-ideal X in a Banach space Y in some way expresses that the norm of Y is a sort of maximum norm (hence the letter M). To be more precise, a subspace X of a Banach space Y is an M-ideal in Y if its annihilator, $X^\perp$, is ℓ_1-complemented (then we may write $Y^* = X^\perp \oplus_1 X^*$, see Section 1).

As it is quoted in the book written by Harmand, Werner and Werner [28]: "The fact that X is an M-ideal in Y has a strong impact on both Y and X since there are a number of important properties shared by M-ideals, but not by arbitrary subspaces". One of the interesting properties shared by M-ideals is the following: if X is an M-ideal in Y then every linear functional defined in X has a *unique* norm preserving extension to a functional in Y^* [28, Proposition I.1.12].

As a consequence of [28, Proposition VI.3.1] we know that if X is an M-ideal in Y, then $\widehat{\bigotimes}_{\varepsilon_k}^{k} X$ (the k-th tensor product of X endowed with the injective tensor norm) is an M-ideal in $\widehat{\bigotimes}_{\varepsilon_k}^{k} Y$. Therefore, every integral k-linear form in X has a unique extension to Y that preserves the integral norm.

Most of the results of the theory of tensor products and tensor norms have their natural analogue in the symmetric context (i.e. in the theory of symmetric tensor products); so one should expect that whenever X is a nontrivial M-ideal in Y, then $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X$ (the k-th symmetric tensor product of X endowed with the injective symmetric tensor norm) would be an M-ideal in $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} Y$. Surprisingly, we will see in Theorem 2.5 that, for real Banach spaces, this never can happen. To prove this, we make use of a characterization of the extreme points of the unit ball of the space of integral polynomials over real Banach spaces, which is interesting in its own right.

In [8] Boyd and Lassalle proved that if X is a real Banach space, X^* has the approximation property and $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X$ does not contain a copy of ℓ_1, then the set of extreme points of the unit ball of the space of integral k-homogeneous polynomials over X is $\{\pm\phi^k : \phi \in X^*, \ \|\phi\| = 1\}$. We will show in Theorem 2.1 that all the additional hypotheses of their result can be removed.

Even though the M-structure for symmetric tensors fails, one may wonder whether the consequence about unique norm preserving extensions holds. That is, being X a nontrivial M-ideal in Y, has every integral k-homogeneous polynomial in X a unique extension to Y that preserves the integral norm? We will give in Theorem 2.10 a positive answer for the case of X being an Asplund space and describe explicitly this unique extension. In particular, if X is an M-ideal in its bidual X^{**} then every integral k-homogeneous polynomial in X has a unique extension to X^{**} with the same integral norm.

We will also examine the following related question: Let $\mathcal{Q}$ be a maximal polynomial ideal and let P be a fixed polynomial belonging to $\mathcal{Q}(^k X)$, under what conditions do we have a *unique* norm preserving extension of P to the bidual X^{**}? Since the Aron–Berner extension preserves the ideal norm for maximal polynomial ideals [15], the question can be rephrased in the following way: When is the Aron–Berner extension the *only* norm preserving extension (for a given polynomial) in $\mathcal{Q}$?

This was addressed in [5] for the ideal of continuous homogeneous polynomials. We will see in Section 4 necessary and sufficient conditions for this to happen that are related with the continuity of the Aron–Berner extension morphism.

To this end, given a symmetric tensor norm of order k, β_k, we examine the local geometry of the bidual of the symmetric tensor product of a Banach space, $(\widehat{\bigotimes}_{\beta_k}^{k,s} X)^{**}$, and the symmetric tensor product of its bidual $\widehat{\bigotimes}_{\beta_k}^{k,s} X^{**}$. We introduce a canonical application Θ_{β_k} from $\widehat{\bigotimes}_{\beta_k}^{k,s} X^{**}$ to $(\widehat{\bigotimes}_{\beta_k}^{k,s} X)^{**}$, in order to study if these spaces have the same local structure in the sense of the principle of local reflexivity. Some isomorphic properties are derived from this relationship. For example, in Theorem 3.3 we show that, if X^{**} has the bounded approximation property, then Θ_{β_k} embeds $\widehat{\bigotimes}_{\beta_k}^{k,s} X^{**}$ as a locally complemented subspace of $(\widehat{\bigotimes}_{\beta_k}^{k,s} X)^{**}$. Equivalently, $\mathcal{P}_{\beta_k}(^k X^{**})$ (the maximal ideal of β_k-continuous k-homogeneous polynomials over X^{**}) is a complemented subspace of $\mathcal{P}_{\beta_k}(^k X)^{**}$. This extends results of Jaramillo–Prieto–Zalduendo [29, Corollary 3] and Cabello–García [10, Theorem 2].

We will also find conditions to ensure the existence of a canonical isomorphism between these two spaces, i.e. the Q-reflexivity for the β_k norm introduced by Aron and Dineen [6].

The article is organized as follows. In Section 1 we give some preliminary background. All the results mentioned above regarding the injective symmetric tensor norm and the ideal of integral polynomials are presented in Section 2. In Section 3 we describe and study some properties of the mapping Θ_{β_k} which relates $\widehat{\bigotimes}_{\beta_k}^{k,s} X^{**}$ and $(\widehat{\bigotimes}_{\beta_k}^{k,s} X)^{**}$. In Section 4 we study necessary and sufficient conditions that assure that a given k-homogeneous polynomial P belonging to a maximal polynomial ideal $\mathcal{Q}(^k X)$ has a unique norm preserving extension to $\mathcal{Q}(^k X^{**})$. In Section 5 we investigate under which circumstances the mapping Θ_{β_k} (equivalently $(\Theta_{\beta_k})^*$) becomes an isomorphism; providing thus an isomorphism between $\mathcal{P}_{\beta_k}(^k X)^{**}$ and $\mathcal{P}_{\beta_k}(^k X^{**})$.

We refer to [23,25] for the background about symmetric tensor products and polynomial ideals and to [28] for the theory of M-ideals.

1. Preliminaries

Throughout the paper X and Y will be real or complex Banach spaces, X^* will denote the dual space of X, B_X will be the closed unit ball of X and S_X will stand for the unit sphere. The canonical inclusion from X to X^{**} will be denoted by κ_X. We will also note by $FIN(X)$ the class of all finite dimensional subspaces of X.

We will use the notation $\bigotimes^k X$ for the k-fold tensor product of X. For simplicity, $\otimes^k x$ will stand for the elementary tensor $x \otimes \overset{k}{\cdots} \otimes x$. The subspace of $\bigotimes^k X$ consisting of all tensors of the form $\sum_{j=1}^r \lambda_j \otimes^k x_j$, where λ_j is a scalar and $x_j \in X$ for all j, is called the *symmetric k-fold tensor product of X* and is denoted by $\bigotimes^{k,s} X$. When X is a vector space over $\mathbb{C}$, the scalars are not needed in the previous expression.

Given a continuous operator $T: X \to Y$, *the symmetric k-tensor power of T* (or the *tensor operator of T*) is the mapping from $\bigotimes^{k,s} X$ to $\bigotimes^{k,s} Y$ defined by

$$\left(\otimes^{k,s} T\right)\left(\otimes^k x\right) = \otimes^k (Tx)$$

on the elementary tensors and extended by linearity.

For a k-fold symmetric tensor $v \in \bigotimes^{k,s} X$, the *symmetric projective norm* of v is given by

$$\pi_{k,s}(v) = \inf\left\{\sum_{j=1}^r |\lambda_j| \|x_j\|^k\right\},$$

where the infimum is taken over all the representations of v of the form $\sum_{j=1}^r \lambda_j \otimes^k x_j$.

On the other hand, *the symmetric injective norm* of v is defined by

$$\varepsilon_{k,s}(v) = \sup_{\phi \in B_{X^*}} \left|\sum_{j=1}^r \lambda_j \phi(x_j)^k\right|,$$

where $\sum_{j=1}^r \lambda_j \otimes^k x_j$ is any fixed representation of v. For properties of these two classical norms ($\varepsilon_{k,s}$ and $\pi_{k,s}$) see [23].

Symmetric tensor products linearize homogeneous polynomials. Recall that a function $P: X \to \mathbb{K}$ is said to be a (continuous) *k-homogeneous polynomial* if there exists a (continuous) symmetric k-linear form

$$A: X \times \overset{k}{\cdots} \times X \to \mathbb{K}$$

such that $P(x) = A(x, \ldots, x)$ for all $x \in X$. In this case, A is called the symmetric k-linear form associated to P and it is usually denoted by $\overset{\vee}{P}$. Continuous k-homogeneous polynomials are those bounded in the unit ball, and the norm of such P is given by

$$\|P\| = \sup_{\|x\| \leqslant 1} |P(x)|.$$

If we denote by $\mathcal{P}(^k X)$ the Banach space of all continuous k-homogeneous polynomials on X endowed with the sup norm, we have the isometric identification

$$\mathcal{P}(^k X) \overset{1}{=} \left(\bigotimes_{\pi_{k,s}}^{k,s} X\right)^*. \tag{1}$$

We say that β_k is a *symmetric tensor norm of order k* (s-tensor norm) if β_k assigns to each normed space X a norm $\beta_k(.; \bigotimes^{k,s} X)$ on the k-fold symmetric tensor product $\bigotimes^{k,s} X$ such that

(1) $\varepsilon_{k,s} \leqslant \beta_k \leqslant \pi_{k,s}$ on $\bigotimes^{k,s} X$.

(2) $\| \otimes^{k,s} T : \bigotimes_{\beta_k}^{k,s} X \to \bigotimes_{\beta_k}^{k,s} Y \| \leqslant \|T\|^k$ for each operator $T \in \mathcal{L}(X, Y)$.

Condition (2) will be referred to as the "*metric mapping property*". We denote by $\bigotimes_{\beta_k}^{k,s} X$ the tensor product $\bigotimes^{k,s} X$ endowed with the norm $\beta_k(.; \bigotimes^{k,s} X)$, and we write $\widehat{\bigotimes}_{\beta_k}^{k,s} X$ for its completion.

An s-tensor norm β_k is called *finitely generated* if for every normed space X and $v \in \bigotimes^{k,s} X$, we have:

$$\beta_k \left(v, \bigotimes^{k,s} X \right) = \inf \left\{ \beta_k \left(v, \bigotimes^{k,s} M \right) : M \in FIN(X), \ v \in \bigotimes^{k,s} M \right\}.$$

From now on, every s-tensor norm considered in the article will be finitely generated.

If β_k is an s-tensor norm of order k, then the *dual tensor norm* β_k^* is defined on FIN (the class of finite dimensional spaces) by

$$\bigotimes_{\beta_k^*}^{k,s} M \overset{1}{:=} \left(\bigotimes_{\beta_k}^{k,s} M^* \right)^* \tag{2}$$

and on NORM (the class of normed spaces) by

$$\beta_k^* \left(v, \bigotimes^{k,s} X \right) := \inf \left\{ \beta_k^* \left(v, \bigotimes^{k,s} M \right) : v \in \bigotimes^{k,s} M \right\},$$

the infimum being taken over all the finite dimensional subspaces M of X whose symmetric tensor product contains v.

Since any s-tensor norm satisfies $\beta_k \leqslant \pi_{k,s}$, we have a dense inclusion

$$\bigotimes_{\beta_k}^{k,s} X \hookrightarrow \bigotimes_{\pi_{k,s}}^{k,s} X.$$

As a consequence, any $P \in (\bigotimes_{\beta_k}^{k,s} X)^*$ can be thought as a k-homogeneous polynomial on X. Different s-tensor norms β_k give rise, by this duality, to different classes of polynomials.

We will say that β_k is *projective* if, for every metric surjection $Q : X \overset{1}{\twoheadrightarrow} Y$, the tensor product operator

$$\otimes^{k,s} Q : \bigotimes_{\beta_k}^{k,s} X \to \bigotimes_{\beta_k}^{k,s} Y$$

is also a metric surjection. On the other hand we will say that α is *injective* if, for every $I : X \overset{1}{\hookrightarrow} Y$ isometric embedding, the tensor product operator

$$\otimes^{k,s} I : \bigotimes_{\beta_k}^{k,s} X \to \bigotimes_{\beta_k}^{k,s} Y,$$

is an isometric embedding.

The two extreme s-tensor norms, π_s and ε_s, are examples of the last two definition: π_s is projective and ε_s is injective.

The projective and injective associates (or hulls) of a symmetric tensor norm β_k will be denoted, by extrapolation of the 2-fold full case, as $\backslash\beta_k/$ and $/\beta_k\backslash$ respectively. The projective associate of β_k will be the (unique) smallest projective tensor norm greater than β_k. Following some ideas from [19, Theorem 20.6] we have

$$\otimes^{k,s} Q_X : \bigotimes_{\beta_k}^{k,s} \ell_1(X) \overset{1}{\twoheadrightarrow} \bigotimes_{\backslash\beta_k/}^{k,s} X,$$

where $Q_X : \ell_1(B_X) \twoheadrightarrow X$ is the canonical quotient map. We say that β_k is projective if $\beta_k = \backslash\beta_k/$.

The injective associate of β_k will be the (unique) greatest injective tensor norm smaller than β_k. As in [19, Theorem 20.7] we get,

$$\otimes^{k,s} I_X : \bigotimes_{/\beta_k\backslash}^{k,s} X \overset{1}{\hookrightarrow} \bigotimes_{\beta_k}^{k,s} \ell_\infty(B_{X^*}),$$

where I_X is the canonical embedding. We say that β_k is injective if $\beta_k = /\beta_k\backslash$.

The following duality relations for an s-tensor norm β_k are easily obtained (see [16])

$$\left(/\beta_k\backslash\right)^* = \backslash\beta_k^*/, \qquad \left(\backslash\beta_k/\right)^* = /\beta_k^*\backslash.$$

Let us recall some definitions on the theory of Banach polynomial ideals [24]. A *Banach ideal of continuous scalar valued k-homogeneous polynomials* is a pair $(\mathcal{Q}, \|\cdot\|_\mathcal{Q})$ such that:

  (i) $\mathcal{Q}(^kX) = \mathcal{Q} \cap \mathcal{P}(^kX)$ is a linear subspace of $\mathcal{P}(^kX)$ and $\|\cdot\|_{\mathcal{Q}(^kX)}$ (the restriction of $\|\cdot\|_\mathcal{Q}$ to $\mathcal{Q}(^kX)$) is a norm which makes $(\mathcal{Q}(^kX), \|\cdot\|_{\mathcal{Q}(^kX)})$ a Banach space.
 (ii) If $T \in \mathcal{L}(X_1, X)$, $P \in \mathcal{Q}(^kX)$ then $P \circ T \in \mathcal{Q}(^kX_1)$ and

$$\|P \circ T\|_{\mathcal{Q}(^kX_1)} \leqslant \|P\|_{\mathcal{Q}(^kX)} \|T\|^k.$$

(iii) $z \mapsto z^k$ belongs to $\mathcal{Q}(^k\mathbb{K})$ and has norm 1.

Let $(\mathcal{Q}, \|\cdot\|_\mathcal{Q})$ be a Banach ideal of continuous scalar valued k-homogeneous polynomials and, for $P \in \mathcal{P}(^kX)$, define

$$\|P\|_{\mathcal{Q}^{max}(^kX)} := \sup\{\|P|_M\|_{\mathcal{Q}(^kM)} : M \in FIN(X)\} \in [0, \infty].$$

The *maximal hull* of $\mathcal{Q}$ is the ideal given by $\mathcal{Q}^{max} := \{P \in \mathcal{P}(^kX) : \|P\|_{\mathcal{Q}^{max}} < \infty\}$. An ideal $\mathcal{Q}$ is said to be *maximal* if $\mathcal{Q} \overset{1}{=} \mathcal{Q}^{max}$. The *minimal hull* of $\mathcal{Q}$ is the composition ideal $\mathcal{Q}^{min} := \mathcal{Q} \circ \overline{\mathcal{F}}$, where $\overline{\mathcal{F}}$ stands for the ideal of approximable operators, with the usual composition norm. An ideal $\mathcal{Q}$ is said to be *minimal* if $\mathcal{Q} \overset{1}{=} \mathcal{Q}^{min}$.

By [25], a maximal (scalar-valued) ideal of k-homogeneous polynomials is dual to a symmetric tensor product endowed with a finitely generated s-tensor norm of order k. So, for β_k a finitely generated s-tensor norm of order k, we denote by $\mathcal{P}_{\beta_k}$ the polynomial ideal dual to this tensor norm. That is, for a Banach space X,

$$\mathcal{P}_{\beta_k}(^kX) = \left(\widehat{\bigotimes}_{\beta_k}^{k,s} X\right)^*.$$

The maximal ideal dual to the symmetric tensor norm $\beta_k = \varepsilon_{k,s}$ is the ideal of integral polynomials $\mathcal{P}_I$. A polynomial $P \in \mathcal{P}(^kX)$ is *integral* if there exists a regular Borel measure μ, of bounded variation on (B_{X^*}, w^*) such that

$$P(x) = \int\limits_{B_{X^*}} \phi(x)^k \, d\mu(\phi),$$

for all $x \in X$. The integral norm of P is given by

$$\|P\|_{\mathcal{P}_I(^kX)} = \inf\{|\mu|(B_{X^*})\},$$

where the infimum is taken over all the measures μ representing P as above.

The minimal hull of $\mathcal{P}_I$ is the ideal of nuclear polynomials $\mathcal{P}_N$. A polynomial $P \in \mathcal{P}(^kX)$ is *nuclear* if it can be written as $P(x) = \sum_{j=1}^{\infty} \lambda_j \phi_j(x)^k$, where $\lambda_j \in \mathbb{K}$, $\phi_j \in X^*$ for all j and $\sum_{j=1}^{\infty} |\lambda_j| \|\phi_j\|^k < \infty$. The nuclear norm of P is

$$\|P\|_{\mathcal{P}_N(^kX)} = \inf\left\{ \sum_{j=1}^{\infty} |\lambda_j| \|\phi_j\|^k \right\},$$

where the infimum is taken over all the representations of P as above.

Aron and Berner showed in [4] how to extend continuous polynomials defined on a Banach space X to the bidual X^{**}. Given a continuous k-homogeneous polynomial $P : X \to \mathbb{K}$ the 'Aron–Berner' extension $\overline{P}$ of P is given by means of the corresponding symmetric k-linear form A. For a k-tuple $(z_1, \ldots, z_k) \in X^{**} \times \cdots \times X^{**}$, consider

$$\overline{A}(z_1, \ldots, z_k) := \lim_{i_1} \cdots \lim_{i_k} A(x_{i_1}, \ldots, x_{i_k}),$$

where each (x_{i_j}) is a net in X which converges to z_j in the weak* topology ($j = 1, \ldots, k$). The *Aron–Berner extension* of P is defined by

$$\overline{P}(z) := \overline{A}(z, \ldots, z).$$

Although the definition of $\overline{A}(z_1, \ldots, z_k)$ depends on the order in which one calculates the limits, the definition of the extended polynomial $\overline{P}$ is independent of the order used (see [21,41,42] for further details and properties of this extension).

In 1989 Davie and Gamelin [18] proved that the Aron–Berner extension preserves the uniform norm. In other words, they proved that this extension is a 'real' Hahn–Banach extension. Recently, Carando and the second author [15] extended this result for maximal and minimal polynomial ideals. More precisely, they showed that every maximal or minimal ideal of k-homogeneous polynomials $\mathcal{Q}$ is closed under the Aron–Berner extension (i.e. for every Banach space X and every polynomial P in $\mathcal{Q}(^kX)$, the Aron–Berner extension $\overline{P}$ is in $\mathcal{Q}(^kX^{**})$). Moreover, the Aron–Berner extension morphism $AB : \mathcal{Q}(^kX) \to \mathcal{Q}(^kX^{**})$ given by $P \mapsto \overline{P}$ is an isometry for every Banach space X:

$$\|P\|_{\mathcal{Q}(^kX)} = \|\overline{P}\|_{\mathcal{Q}(^kX^{**})}. \tag{3}$$

Given a polynomial ideal $\mathcal{Q}$ closed under the Aron–Berner extension and a continuous linear morphism $s : X^* \to Y^*$, we can construct the following mapping $\bar{s} : \mathcal{Q}(^k X) \to \mathcal{Q}(^k Y)$ given by

$$\bar{s}(P) := \overline{P} \circ s^* \circ \kappa_Y,$$

where $\kappa_Y : Y \to Y^{**}$ is the canonical inclusion. The mapping $\bar{s}$ is referred as 'the extension morphism' of s. If s is an isomorphism then $\bar{s}$ is also an isomorphism. Moreover, if s is an isometry and the Aron–Berner extension morphism $AB : \mathcal{Q}(^k X) \to \mathcal{Q}(^k X^{**})$ is an isometric mapping (for example if $\mathcal{Q}$ is maximal or minimal) then it is easy to see that $\bar{s}$ is also an isometry. For more properties and details about $\bar{s}$ see [31,41].

Recall that a Banach space X is said to have the *approximation property* (AP) if, for every absolutely convex compact set K and every $\varepsilon > 0$ there is a finite rank operator $T \in \mathcal{L}(X, X)$ with $\|Tx - x\| < \varepsilon$ for all $x \in K$.

The bounded approximation property is a version of this property with control of the norms of the finite rank operators involved. It can be defined equivalently in the following way. The space X is said to have the λ-*approximation property* (λ-AP) if there is a net T_γ of finite rank operators, with $\|T_\gamma\| \leqslant \lambda$, such that $\lim_\gamma \|T_\gamma(x) - x\| = 0$ for all $x \in X$. A space having the λ-AP for some finite λ is said to have the *bounded approximation property* (BAP). Also, the 1-AP is called *metric approximation property* (MAP).

A closed subspace X of a Banach space Y is an *M-ideal* in Y if $Y^* = X^\perp \oplus_1 X^\sharp$, where $X^\perp$ is the annihilator of X and $X^\sharp$ is a closed subspace of Y^*. Since $X^\sharp$ can be (isometrically) identified with X^*, it is usual to denote $Y^* = X^\perp \oplus_1 X^*$. However, we often prefer to state explicitly the isometric mapping $s : X^* \to Y^*$, thus obtaining the decomposition $Y^* = X^\perp \oplus_1 s(X^*)$. The space X is said to be *M-embedded* if X is an M-ideal in its bidual X^{**}.

A Banach space X is *Asplund* (or a strong differentiability space) if every separable subspace S of X has separable continuous dual space S^*, or equivalently if its dual space X^* has the Radon–Nikodým property [21,28]. Recall that every M-embedded space is Asplund [28, Theorem III.3.1].

A point $x \in B_X$ is said to be a real (complex) *extreme point* whenever $\{x + \zeta y : |\zeta| \leqslant 1, \zeta \in \mathbb{R}\} \subset B_X$ for $y \in X$ implies $y = 0$ (respectively $\zeta \in \mathbb{C}$). In complex Banach spaces, it is easy to check that every real extreme point of B_X is also a complex extreme point. The converse however is not true, since, for instance, every point of S_{ℓ_1} is a complex extreme point of B_{ℓ_1}. We denote by $Ext(B_X)$ the set of real extreme points of the ball B_X. When X is an M-ideal in Y, we have the following equality for the sets of extreme points of the unit balls [28, Lemma I.1.5]:

$$Ext(B_{Y^*}) = Ext(B_{X^\perp}) \cup Ext(B_{X^*}).$$

2. Integral polynomials on *M*-ideals

If X is an M-ideal in Y then, by [28, Proposition VI.3.1], the associativity of the ε-norm and the transitivity of M-ideals, it results that $\widehat{\bigotimes}_{\varepsilon_k}^k X$ is an M-ideal in $\widehat{\bigotimes}_{\varepsilon_k}^k Y$. This clearly implies that any k-linear integral form on X (being an element of the dual of $\widehat{\bigotimes}_{\varepsilon_k}^k X$) has a unique (integral) norm preserving extension to a k-linear integral form on Y.

The intuition leads us to think that the same happens in the symmetric case. That is, if X is an M-ideal in Y then $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X$ should be an M-ideal in $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} Y$ and any integral k-homogeneous polynomial on X should have a unique (integral) norm preserving extension to an integral k-homogeneous polynomial on Y. But things are not always as we expected them to be.

We will see in this section how some properties could change completely and others remain the same in the symmetric case. For real Banach spaces the M-ideal condition is *never* (except trivial cases) maintained for symmetric injective tensor products. On the other hand, on M-embedded spaces, both in the real and complex settings, the unique norm preserving extension of integral polynomials holds.

The negative result for real Banach spaces derives from a characterization of the extreme points of integral polynomials which is interesting by itself. We make use of some results and ideas from [9] and [8], pushing things a little more to obtain a general statement.

In [9] Boyd and Ryan investigated the set of extreme points of the unit ball of $\mathcal{P}_I(^kX)$, for $k > 1$, and they showed the following facts:

(a) For a real Banach space X, $\{\pm\phi^k \colon \phi \in S_{X^*} \text{ and } \phi \text{ attains its norm}\} \subseteq Ext(B_{\mathcal{P}_I(^kX)})$.
(b) For a real or complex Banach space X, $Ext(B_{\mathcal{P}_I(^kX)}) \subseteq \{\pm\phi^k \colon \phi \in S_{X^*}\}$ (see also [12]).

In [8] Boyd and Lassalle proved that if X is a real Banach space, X^* has the approximation property and $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X$ does not contain a copy of ℓ_1, then $Ext(B_{\mathcal{P}_I(^kX)})$ is in fact the set $\{\pm\phi^k \colon \phi \in S_{X^*}\}$. In the following theorem we show that the hypotheses of their result are not necessary.

Theorem 2.1. *For a real Banach space X and a positive integer $k > 1$, the set of real extreme points of the unit ball of $\mathcal{P}_I(^kX)$ is $\{\pm\phi^k \colon \phi \in S_{X^*}\}$.*

Proof. Let $\phi \in S_{X^*}$. Since it is clear that ϕ is a norm attaining element of $S_{X^{***}}$, by the previous comment (a), ϕ^k is an extreme point of the unit ball of $\mathcal{P}_I(^kX^{**})$.

Inspired by the proof of Lemma 1 of [8], we will use the fact that $Ext(B) \cap A \subseteq Ext(A)$ whenever $A \subseteq B$. Consider the isometric inclusion

$$AB \colon \mathcal{P}_I(^kX) \overset{1}{\hookrightarrow} \mathcal{P}_I(^kX^{**})$$

given by the Aron–Berner extension morphism $P \mapsto \overline{P}$. Thus,

$$Ext(B_{\mathcal{P}_I(^kX^{**})}) \cap B_{\mathcal{P}_I(^kX)} \subseteq Ext(B_{\mathcal{P}_I(^kX)}).$$

Finally,

$$\{\pm\phi^k \colon \phi \in S_{X^*}\} \subseteq Ext(B_{\mathcal{P}_I(^kX^{**})}) \cap B_{\mathcal{P}_I(^kX)} \subseteq Ext(B_{\mathcal{P}_I(^kX)}) \subseteq \{\pm\phi^k \colon \phi \in S_{X^*}\}. \quad \square$$

Remark 2.2. The previous result is not true for complex Banach spaces. Indeed, Dineen [22, Proposition 4.1] proves that, if X is a complex Banach space, then $Ext(B_{\mathcal{P}_I(^kX)})$ is contained in $\{\phi^k \colon \phi \text{ is a complex extreme point of } B_{X^*}\}$. Let us consider X the complex space ℓ_1. It is clear that $\phi = (0, 1, \dots, 1, \dots) \in S_{\ell_\infty}$ is not a complex extreme point of B_{ℓ_∞}. Hence, ϕ^k could not be an extreme point of $B_{\mathcal{P}_I(^k\ell_1)}$.

Remark 2.3. Although the spaces $\mathcal{P}_I(^kX)$ and $\mathcal{L}_I(^kX)$ can be isomorphic (for example if X is stable [3]), they are very different from a geometric point of view since the set $Ext(B_{\mathcal{L}_I(^kX)})$ is equal to $\{\phi_1\phi_2\cdots\phi_k \colon \phi_i \in Ext(B_{X^*})\}$ (see [36,9]).

The last characterization of the extreme points of the ball of integral polynomials leads us to show that for a real Banach space X, $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X$ is never an M-ideal in $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X^{**}$, unless X is reflexive. As we have already said, this is a big difference with what happens in the nonsymmetric case where for X an M-embedded space it follows that $\widehat{\bigotimes}_{\varepsilon_k}^{k} X$ is an M-ideal in $\widehat{\bigotimes}_{\varepsilon_k}^{k} X^{**}$ [28, Proposition VI.3.1].

Theorem 2.4. *If the real Banach space X is not reflexive, then $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X$ is not an M-ideal in $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X^{**}$.*

Proof. Suppose that $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X$ is an M-ideal in $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X^{**}$. Then we would have:

$$Ext(B_{\mathcal{P}_I(^k X^{**})}) = Ext(B_{\mathcal{P}_I(^k X)}) \cup Ext(B_{(\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X)^{\perp}}).$$

By the description of extreme points of integral polynomials of the previous theorem, this equality would imply

$$Ext(B_{(\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X)^{\perp}}) = \{\pm\phi^k \colon \phi \in S_{X^{***}} \setminus S_{X^*}\}.$$

This is not possible since through the decomposition $X^{***} = X^* \oplus X^{\perp}$ if we choose $\phi \in S_{X^{***}}$ such that $\phi = \phi_1 + \phi_2$, with $\phi_1 \in X^*$, $\phi_2 \in X^{\perp}$, $\phi_1, \phi_2 \neq 0$, then $\phi \in S_{X^{***}} \setminus S_{X^*}$ but $\phi^k \notin (\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X)^{\perp}$. $\square$

With almost the same argument (only changing the decomposition $X^{***} = X^* \oplus X^{\perp}$ to $Y^* = X^* \oplus_1 X^{\perp}$) we derive the following:

Theorem 2.5. *If X and Y are real Banach spaces and X is a nontrivial M-ideal in Y, then $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X$ is not an M-ideal in $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} Y$.*

Remark 2.6. As it will be stated in Lemma 4.3, in a maximal ideal of polynomials, the w^*-convergence of a bounded net is equivalent to the pointwise convergence. This implies that if two maximal polynomial ideals have the same set of extreme points of the unit balls, then they are the same ideal (isometrically). So, from Theorem 2.1, if $\mathcal{Q}$ is a maximal ideal of k-homogeneous polynomials that satisfies that, on a real Banach space X, the set of extreme points of its unit ball is $\{\pm\phi^k \colon \phi \in S_{X^*}\}$, then it should be $\mathcal{Q}(^k X) = \mathcal{P}_I(^k X)$.

We have no idea how symmetric tensor products on M-ideals behave in the complex setting.

Question 2.7. *If X is a nontrivial M-ideal in Y, X and Y complex Banach spaces, could $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X$ be an M-ideal in $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} Y$?*

Aron, Boyd and Choi [5, Proposition 7] prove that if X is an M-ideal in X^{**} then the Aron–Berner extension is the unique norm preserving extension from $\mathcal{P}_N(^k X)$ to $\mathcal{P}_N(^k X^{**})$. Their argument can be easily adapted to the situation of X being an M-ideal in Y. Recall that in this

case the natural inclusion $s : X^* \to Y^*$ induces a canonical isometry $\bar{s} : \mathcal{P}_N(^k X) \to \mathcal{P}_N(^k Y)$ (see the explanation at the end of the previous section).

Proposition 2.8. *Let X be an M-ideal in Y and let $s : X^* \to Y^*$ be the associated isometric inclusion. For each $P \in \mathcal{P}_N(^k X)$, $\bar{s}(P)$ is the unique norm preserving extension to $\mathcal{P}_N(^k Y)$.*

We want to prove a similar statement for integral polynomials. If X is an Asplund space (which always holds when X is an M-ideal in X^{**}) we will have a positive result. In this case, nuclear and integral polynomials over X coincide isometrically [9,12]. So, by the previous proposition, there is only one nuclear norm preserving extension to Y. But if Y is not Asplund we could presumably have integral nonnuclear extensions of the same integral norm. We will show that this is impossible to happen.

The result is a consequence of the following lemma.

Lemma 2.9. *Let X be an Asplund space which is a subspace of a Banach space Y. Let $Q \in \mathcal{P}_I(^k Y)$. Given $\varepsilon > 0$ there exists $\widetilde{Q} \in \mathcal{P}_N(^k Y)$ such that Q and $\widetilde{Q}$ coincide on X and*

$$\|\widetilde{Q}\|_{\mathcal{P}_N(^k Y)} \leqslant \|Q\|_{\mathcal{P}_I(^k Y)} + \varepsilon.$$

Proof. Since the restriction of Q to X is nuclear, we can take sequences $(\phi_j)_j \subset X^*$ and $(\lambda_j)_j \subset \mathbb{K}$ such that $Q|_X = \sum_{j=1}^{\infty} \lambda_j \phi_j^k$ and

$$\sum_{j=1}^{\infty} |\lambda_j| \|\phi_j\|^k \leqslant \|Q|_X\|_{\mathcal{P}_N(^k X)} + \varepsilon = \|Q|_X\|_{\mathcal{P}_I(^k X)} + \varepsilon \leqslant \|Q\|_{\mathcal{P}_I(^k Y)} + \varepsilon.$$

For each j, let $\widetilde{\phi}_j$ be a Hahn–Banach extension of ϕ_j to Y. If we define $\widetilde{Q} = \sum_{j=1}^{\infty} \lambda_j \widetilde{\phi}_j^k$, then $\widetilde{Q}$ coincides with Q in X and

$$\|\widetilde{Q}\|_{\mathcal{P}_N(^k Y)} \leqslant \sum_{j=1}^{\infty} |\lambda_j| \|\widetilde{\phi}_j\|^k = \sum_{j=1}^{\infty} |\lambda_j| \|\phi_j\|^k \leqslant \|Q\|_{\mathcal{P}_I(^k Y)} + \varepsilon. \qquad \square$$

Theorem 2.10. *Let X be an Asplund space which is an M-ideal in a Banach space Y and let $s : X^* \to Y^*$ be the associated isometric inclusion. If $P \in \mathcal{P}_I(^k X)$ then the canonical extension $\bar{s}(P)$ is the unique norm preserving extension to $\mathcal{P}_I(^k Y)$.*

Proof. The argument is modeled on the proof of [5, Proposition 7]. We include all the steps for the sake of completeness.

Let $P \in \mathcal{P}_I(^k X)$ and suppose there exists a norm preserving extension $Q \in \mathcal{P}_I(^k Y)$ different from $\bar{s}(P)$. Pick y a norm one vector in Y such that $0 < \delta = |Q(y) - \bar{s}(P)(y)|$.

Note that $X \oplus [y]$ is an Asplund space since X also is. So, by Lemma 2.9 applied to $X \oplus [y]$, there exists $\widetilde{Q} \in \mathcal{P}_N(^k Y)$ such that Q and $\widetilde{Q}$ coincide on $X \oplus [y]$ and

$$\|\widetilde{Q}\|_{\mathcal{P}_N(^k Y)} \leqslant \|Q\|_{\mathcal{P}_I(^k Y)} + \frac{\delta}{4} = \|P\|_{\mathcal{P}_I(^k X)} + \frac{\delta}{4}.$$

Take a nuclear representation of $\widetilde{Q} = \sum_{j=1}^{\infty} \lambda_j \phi_j^k$ such that $\sum_{j=1}^{\infty} |\lambda_j| \|\phi_j\|^k \leqslant \|P\|_{\mathcal{P}_I(^k X)} + \frac{\delta}{2}$. Since X is an M-ideal in Y each $\phi_j \in Y^*$ can be written as the sum of $s(\phi_j|_X)$ and $\phi_j^{\perp}$. Moreover, $\|\phi_j\| = \|s(\phi_j|_X)\| + \|\phi_j^{\perp}\|$.

Recall that $\widetilde{Q}$ coincides with P on X, thus, for every $x \in X$,

$$P(x) = \sum_{j=1}^{\infty} \lambda_j \big(s(\phi_j|_X)(x) + \phi_j^{\perp}(x)\big)^k = \sum_{j=1}^{\infty} \lambda_j \phi_j|_X(x)^k.$$

Using this, we easily get that $\bar{s}(P) = \sum_{j=1}^{\infty} \lambda_j (s(\phi_j|_X))^k$. Naturally,

$$\|P\|_{\mathcal{P}_I(^k X)} = \|P\|_{\mathcal{P}_N(^k X)} = \big\|\bar{s}(P)\big\|_{\mathcal{P}_N(^k X)} \leqslant \sum_{j=1}^{\infty} |\lambda_j| \|\phi_j|_X\|^k.$$

Now,

$$\begin{aligned}
0 < \delta = \big|Q(y) - \bar{s}(P)(y)\big| &= \big|\widetilde{Q}(y) - \bar{s}(P)(y)\big| \\
&\leqslant \left|\sum_{j=1}^{\infty} \lambda_j \big(s(\phi_j|_X)(y) + \phi_j^{\perp}(y)\big)^k - \lambda_j s(\phi_j|_X)(y)^k\right| \\
&\leqslant \sum_{j=1}^{\infty} |\lambda_j| \sum_{i=1}^{k} \binom{k}{i} \big\|s(\phi_j|_X)\big\|^{k-i} \big\|\phi_j^{\perp}\big\|^i \\
&= \sum_{j=1}^{\infty} |\lambda_j| \big(\big\|s(\phi_j|_X)\big\| + \big\|\phi_j^{\perp}\big\|\big)^k - |\lambda_j| \big\|s(\phi_j|_X)\big\|^k \\
&= \sum_{j=1}^{\infty} |\lambda_j| \|\phi_j\|^k - \sum_{j=1}^{\infty} |\lambda_j| \|\phi_j|_X\|^k \\
&\leqslant \|P\|_{\mathcal{P}_I(^k X)} + \frac{\delta}{2} - \|P\|_{\mathcal{P}_I(^k X)} = \frac{\delta}{2}.
\end{aligned}$$

This is a contradiction. Thus, the result follows. $\quad\square$

Since M-embedded spaces are Asplund we have a neater statement in this case:

Corollary 2.11. *Let X be an M-ideal in X^{**}. If $P \in \mathcal{P}_I(^k X)$ then the Aron–Berner extension $\overline{P}$ is the unique norm preserving extension to $\mathcal{P}_I(^k X^{**})$.*

It is known that on ℓ_{∞} integral and nuclear polynomials do not coincide. Consider thus a nonnuclear polynomial $P \in \mathcal{P}_I(^k \ell_{\infty})$. By the fact that c_0 is an M-ideal in ℓ_{∞} and the previous corollary, we derive that the restriction of P to c_0 should have integral (equivalently, nuclear) norm strictly smaller than the integral norm of P.

As we commented before, if X is Asplund (actually, in the more general case of $\ell_1 \not\hookrightarrow \widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X$) the spaces of nuclear and integral polynomials coincide isometrically. So, this is the case for most of the classical Banach spaces.

The situation is quite different for Banach spaces containing ℓ_1. We will see below that if X is Banach space that contains a subspace isomorphic to ℓ_1 and whose dual has the MAP, then the quotient space $\mathcal{P}_I(^k X)/\mathcal{P}_N(^k X)$ is nonseparable. To see this, first we look at the case of $X = \ell_1$. Using Theorem 3.3 and arguing as in the proof of [11, Proposition 2.4] we can clarify how the containment of $\mathcal{P}_N(^k \ell_1)$ in $\mathcal{P}_I(^k \ell_1)$ is.

Proposition 2.12. *For $k > 1$, the quotient space $\mathcal{P}_I(^k \ell_1)/\mathcal{P}_N(^k \ell_1)$ contains a subspace isometric to ℓ_∞/c_0. Moreover, $\mathcal{P}_N(^k \ell_1)$ is not complemented in a dual space.*

Proof. The approximation property of ℓ_∞ gives us the equality $\mathcal{P}_N(^k \ell_1) = \widehat{\bigotimes}_{\pi_{k,s}}^{k,s} \ell_\infty$. Also, since every k-homogeneous polynomial on c_0 is approximable, we have $\mathcal{P}_I(^k \ell_1) = (\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} \ell_1)^* = (\widehat{\bigotimes}_{\pi_{k,s}}^{k,s} c_0)^{**}$. Thus, from [10, Corollary 7] or Theorem 3.3 in the following section, we obtain that $\mathcal{P}_N(^k \ell_1)$ is a locally 1-complemented subspace of $\mathcal{P}_I(^k \ell_1)$ (see definition in next section). We can thus picture the following exact sequence (the image of each arrow coincides with the kernel of the next one):

$$0 \longrightarrow \mathcal{P}_N(^k \ell_1) = \widehat{\bigotimes}_{\pi_{k,s}}^{k,s} \ell_\infty \xrightarrow{\Theta_{\pi_{k,s}}} \mathcal{P}_I(^k \ell_1) = (\widehat{\bigotimes}_{\pi_{k,s}}^{k,s} c_0)^{**} \xrightarrow{q} \mathcal{P}_I(^k \ell_1)/\mathcal{P}_N(^k \ell_1) \longrightarrow 0$$

where the mapping $\Theta_{\pi_{k,s}}$ (defined in the next section) in this particular case is just the formal inclusion $\mathcal{P}_N(^k \ell_1) \hookrightarrow \mathcal{P}_I(^k \ell_1)$.

Consider the isometric embedding $\delta : \ell_\infty \to \mathcal{P}_I(^k \ell_1)$ given by

$$\phi \longmapsto \left((x_n)_n \in \ell_1 \longmapsto \delta(\phi)(x_n) = \sum_{n=1}^{\infty} \phi(n) x_n^k \right).$$

It is clear that $\delta(c_0) = \delta(\ell_\infty) \cap \mathcal{P}_N(^k \ell_1)$ and that $q \circ \delta$ factorizes through the quotient ℓ_∞/c_0. Then there is an isometric embedding $\ell_\infty/c_0 \hookrightarrow \mathcal{P}_I(^k \ell_1)/\mathcal{P}_N(^k \ell_1)$ that makes commutative the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & c_0 & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/c_0 & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle \delta|_{c_0}} & & \downarrow{\scriptstyle \delta} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{P}_N(^k \ell_1) & \longrightarrow & \mathcal{P}_I(^k \ell_1) & \xrightarrow{q} & \mathcal{P}_I(^k \ell_1)/\mathcal{P}_N(^k \ell_1) & \longrightarrow & 0
\end{array}
$$

Now, the proof of [11, Proposition 2.4] can be adapted easily to our setting to obtain the desired result. $\quad\square$

For a Banach space X whose dual has the MAP, the space of nuclear polynomials $\mathcal{P}_N(^k X)$ coincides with the projective tensor product $\widehat{\bigotimes}_{\pi_{k,s}}^{k,s} X^*$ and it is contained isometrically in the space

of integral polynomials $\mathcal{P}_I({}^k X) = (\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X)^*$ (see [23, DualityTheorem] or [16, Corollary 3.4]). So, it makes sense to consider the quotient $\mathcal{P}_I({}^k X)/\mathcal{P}_N({}^k X)$.

Corollary 2.13. *Let X be a Banach space whose dual has the MAP. If X contains a copy of ℓ_1, then the quotient space $\mathcal{P}_I({}^k X)/\mathcal{P}_N({}^k X)$ is nonseparable.*

Proof. We can suppose with no lost of generality, that ℓ_1 is a subspace of X. Let $i : \ell_1 \to X$ the inclusion. Since nuclear and integral polynomials are extendible, the restriction mappings $R_N : \mathcal{P}_N({}^k X) \to \mathcal{P}_N({}^k \ell_1)$ and $R_I : \mathcal{P}_I({}^k X) \to \mathcal{P}_I({}^k \ell_1)$ are surjective.

By the comment before the corollary, we have metric injections $J_X : \mathcal{P}_N({}^k X) \to \mathcal{P}_I({}^k X)$ and $J_{\ell_1} : \mathcal{P}_N({}^k \ell_1) \to \mathcal{P}_I({}^k \ell_1)$ that allow us to look at the quotient projections $q_X : \mathcal{P}_I({}^k X) \to \mathcal{P}_I({}^k X)/\mathcal{P}_N({}^k X)$ and $q_{\ell_1} : \mathcal{P}_I({}^k \ell_1) \to \mathcal{P}_I({}^k \ell_1)/\mathcal{P}_N({}^k \ell_1)$.

Thus, we have the diagram

$$
\begin{array}{ccccc}
\mathcal{P}_N({}^k X) & \xrightarrow{J_X} & \mathcal{P}_I({}^k X) & \xrightarrow{q_X} & \mathcal{P}_I({}^k X)/\mathcal{P}_N({}^k X) \\
{\scriptstyle R_N}\downarrow & & {\scriptstyle R_I}\downarrow & & {\scriptstyle R}\downarrow \\
\mathcal{P}_N({}^k \ell_1) & \xrightarrow{J_{\ell_1}} & \mathcal{P}_I({}^k \ell_1) & \xrightarrow{q_{\ell_1}} & \mathcal{P}_I({}^k \ell_1)/\mathcal{P}_N({}^k \ell_1)
\end{array}
$$

where the left side square is easily seen to be commutative and the down arrow R is defined to make the right side square commutative also. This fact and the surjectivity of R_I and q_{ℓ_1} imply that the operator R is onto. Since, by the previous proposition, the image of R is not separable the same should be true for its domain $\mathcal{P}_I({}^k X)/\mathcal{P}_N({}^k X)$. $\quad\square$

3. The bidual of a symmetric tensor product

In order to obtain a characterization of polynomials belonging to an ideal that have unique norm preserving extension, we need to relate the bidual of the symmetric tensor product of a Banach space with the symmetric tensor product of its bidual.
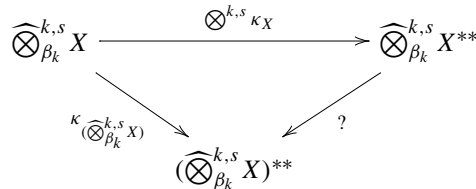
More precisely, for β_k an s-tensor norm of order k, we study the relationship between the spaces $\widehat{\bigotimes}_{\beta_k}^{k,s} X^{**}$ and $(\widehat{\bigotimes}_{\beta_k}^{k,s} X)^{**}$. We need the notion of a local complementation.

Definition 3.1. (See [30].) Let X be a subspace of Y through i. We say that X is locally complemented in Y (through i) if there exists a constant $\lambda > 0$ such that for every finite dimensional subspace $F \subset Y$ there exists and operator $r_F : F \to X$, $\|r_F\| \leqslant \lambda$, such that $r_F(i(x)) = x$ whenever $i(x) \in F$. For the quantitative version we will say locally λ-complemented.

It is known that X is locally complemented in Y (through $i : X \hookrightarrow Y$) if and only if $i^* : Y^* \to X^*$ has a left-inverse of bound λ (i.e. X^* is λ-complemented in Y^*). The Principle of Local Reflexivity of Lindenstrauss and Rosenthal [33] says that every Banach space is locally complemented in its bidual. Also, it is well known that every Banach space is locally complemented in its ultrapowers.

In this terminology, it is clear that the Aron–Berner extension ensures that the subspace $\widehat{\bigotimes}_{\beta_k}^{k,s} X$ is locally complemented in $\widehat{\bigotimes}_{\beta_k}^{k,s} X^{**}$ (through $\otimes^{k,s} \kappa_X : \widehat{\bigotimes}_{\beta_k}^{k,s} X \hookrightarrow \widehat{\bigotimes}_{\beta_k}^{k,s} X^{**}$), then these spaces have the same local structure. Some questions arise:

- Do $\widehat{\bigotimes}^{k,s}_{\beta_k} X^{**}$ and $(\widehat{\bigotimes}^{k,s}_{\beta_k} X)^{**}$ have the same local structure? The answer is yes if X^{**} has the bounded approximation property (Theorem 3.3).
- Is there a mapping that makes commutative the following diagram?

$$
\begin{array}{ccc}
\widehat{\bigotimes}^{k,s}_{\beta_k} X & \xrightarrow{\;\otimes^{k,s}\kappa_X\;} & \widehat{\bigotimes}^{k,s}_{\beta_k} X^{**} \\[2mm]
& \searrow^{\kappa_{(\widehat{\bigotimes}^{k,s}_{\beta_k} X)}} \quad \swarrow_{?} & \\[2mm]
& (\widehat{\bigotimes}^{k,s}_{\beta_k} X)^{**} &
\end{array}
$$

The answer is always yes and the canonical mapping is given by

$$
\Theta_{\beta_k} : \widehat{\bigotimes}^{k,s}_{\beta_k} X^{**} \longrightarrow \left(\widehat{\bigotimes}^{k,s}_{\beta_k} X \right)^{**} = \mathcal{P}_{\beta_k}(^k X)^*
$$

$$
v \longmapsto \big(P \mapsto \langle \overline{P}, v \rangle \big).
$$

Since the Aron–Berner extension preserves the ideal norm for maximal polynomial ideals [15, Corollary 3.4], it is clear that $\|\Theta_{\beta_k}\| = 1$. We want to study when Θ_{β_k} is an isomorphic embedding. As often happens when dealing with tensor products, approximation properties will play a crucial role in our proofs. We adapt some techniques developed in [10] for the projective full tensor product $\widehat{\bigotimes}^{k}_{\pi_k} X$ to the case of a general s-tensor norm β_k.

Cabello and the third author proved in [10] that if X is a Banach space whose bidual has the BAP, then Θ_π embeds $\widehat{\bigotimes}^{k}_{\pi_k} X^{**}$ as a locally complemented subspace of $(\widehat{\bigotimes}^{k}_{\pi_k} X)^{**}$. Equivalently, $\mathcal{L}(^k X^{**})$ is a complemented subspace of $\mathcal{L}(^k X)^{**}$ for all $k \geqslant 1$.

We propose here a version of this result for symmetric tensor products endowed with any s-tensor norm β_k. To prove it, we need first a lemma.

Lemma 3.2. *Let β_k be an s-tensor norm of order k. If X^{**} has the λ-AP, then $\widehat{\bigotimes}^{k,s}_{\beta_k} X^{**}$ has the λ^k-AP. Moreover, there is a net of finite rank operators $(t_\gamma)_\gamma$ with $t_\gamma : X \to X^{**}$ such that the operators $T_\gamma = \otimes^{k,s} t_\gamma^{**} : \widehat{\bigotimes}^{k,s}_{\beta_k} X^{**} \to \widehat{\bigotimes}^{k,s}_{\beta_k} X^{**}$ satisfy $\|T_\gamma\| \leqslant \lambda^k$ and*

$$
\lim_\gamma \beta_k \big(T_\gamma(v) - v \big) = 0, \quad \text{for all } v \in \widehat{\bigotimes}^{k,s}_{\beta_k} X^{**}.
$$

Proof. By [10, Corollary 1], the λ-AP of X^{**} can be realized by mappings t_γ^{**} where $t_\gamma : X \to X^{**}$ are finite rank operators satisfying $\|t_\gamma\| \leqslant \lambda$. This implies that the operators T_γ have finite rank and $\|T_\gamma\| = \|t_\gamma^{**}\|^k \leqslant \lambda^k$.

Now, to see that the net $\{T_\gamma\}_\gamma$ approximates the identity, let us begin with an elementary tensor $\otimes^k z \in \bigotimes^{k,s}_{\beta_k} X^{**}$. We have,

$$
\beta_k \big(T_\gamma(\otimes^k z) - \otimes^k z \big) \leqslant \pi_{k,s} \big(T_\gamma(\otimes^k z) - \otimes^k z \big)
$$

$$
= \sup_{P \in B_{\mathcal{P}(^k X^{**})}} \big| \langle P, T_\gamma(\otimes^k z) - \otimes^k z \rangle \big|
$$

$$= \sup_{P \in B_{\mathcal{P}(^k X^{**})}} \left| P\left(t_\gamma^{**} z\right) - P(z) \right|$$

$$= \sup_{P \in B_{\mathcal{P}(^k X^{**})}} \left| \sum_{j=0}^{k-1} \binom{k}{j} \check{P}\left(z^j, \left(t_\gamma^{**} z - z\right)^{k-j}\right) \right|$$

$$\leqslant \sup_{P \in B_{\mathcal{P}(^k X^{**})}} \sum_{j=0}^{k-1} \binom{k}{j} \|\check{P}\| \cdot \|z\|^j \cdot \left\| t_\gamma^{**} z - z \right\|^{k-j} \longrightarrow 0.$$

As a consequence, for any finite sum $u = \sum_{i=1}^N \lambda_i \otimes^k z_i$ it follows that $\beta_k(T_\gamma(u) - u) \to 0$. Finally, since an arbitrary $v \in \widehat{\bigotimes}_{\beta_k}^{k,s} X^{**}$ can be approximated by finite sums of elementary tensors and the norms of the operators T_γ are bounded, we get that $\beta_k(T_\gamma(v) - v) \to 0$. $\quad\square$

Theorem 3.3. *Let β_k be an s-tensor norm of order k. If X^{**} has the λ-AP, then Θ_{β_k} embeds $\widehat{\bigotimes}_{\beta_k}^{k,s} X^{**}$ as a locally λ^k-complemented subspace of $(\widehat{\bigotimes}_{\beta_k}^{k,s} X)^{**}$. Equivalently, $\mathcal{P}_{\beta_k}(^k X^{**})$ is a λ^k-complemented subspace of $\mathcal{P}_{\beta_k}(^k X)^{**}$.*

Proof. We have already observed that $\|\Theta_{\beta_k}(v)\| \leqslant \beta_k(v)$ for all $v \in \widehat{\bigotimes}_{\beta_k}^{k,s} X^{**}$. Consider, as in Lemma 3.2, a net of finite rank operators $t_\gamma : X \to X^{**}$. Thus, the net $T_\gamma = \otimes^{k,s} t_\gamma^{**}$ transfers the BAP to $\widehat{\bigotimes}_{\beta_k}^{k,s} X^{**}$, with constant λ^k. In view of [10, Lemma 4], the proof will be completed if we show that for each γ there is an operator $\widetilde{T}_\gamma$ making commutative the diagram

$$
\begin{array}{ccc}
\widehat{\bigotimes}_{\beta_k}^{k,s} X^{**} & \xrightarrow{\quad \Theta_{\beta_k} \quad} & (\widehat{\bigotimes}_{\beta_k}^{k,s} X)^{**} \\
& T_\gamma \searrow \quad \swarrow \widetilde{T}_\gamma & \\
& \widehat{\bigotimes}_{\beta_k}^{k,s} X^{**} &
\end{array}
\qquad (4)
$$

Let us consider the finite rank operators

$$\widetilde{T}_\gamma = \left(\otimes^{k,s} t_\gamma\right)^{**} : \left(\widehat{\bigotimes}_{\beta_k}^{k,s} X\right)^{**} \longrightarrow \widehat{\bigotimes}_{\beta_k}^{k,s} X^{**}.$$

Note that the ranges of the operators $\widetilde{T}_\gamma$ are in $\widehat{\bigotimes}_{\beta_k}^{k,s} X^{**}$ instead of $(\widehat{\bigotimes}_{\beta_k}^{k,s} X^{**})^{**}$ because the mappings $\otimes^{k,s} t_\gamma$ have finite rank. It is clear that $\|\widetilde{T}_\gamma\| \leqslant \lambda^k$.

We have the following factorization (see (4)):

$$T_\gamma = \widetilde{T}_\gamma \circ \Theta_{\beta_k}.$$

Indeed, by linearity it is enough to prove that $T_\gamma(\otimes^k z) = \widetilde{T}_\gamma(\Theta_{\beta_k}(\otimes^k z))$, for any $z \in X^{**}$. To see this, let $Q \in \mathcal{P}_{\beta_k}(^k X^{**})$, then

$$\langle Q, \widetilde{T}_\gamma (\Theta_{\beta_k}(\otimes^k z)) \rangle = \langle Q, (\otimes^{k,s} t_\gamma)^{**} (\Theta_{\beta_k}(\otimes^k z)) \rangle = \Theta_{\beta_k}(\otimes^k z) ((\otimes^{k,s} t_\gamma)^*(Q))$$

$$= \Theta_{\beta_k}(\otimes^k z)(Q \circ \otimes^{k,s} t_\gamma) = \overline{Q \circ \otimes^{k,s} t_\gamma}(z)$$

$$= Q(t_\gamma^{**} z) = \langle Q, T_\gamma(\otimes^k z) \rangle. \qquad \square$$

Note that the fact that Θ_{β_k} embeds $\widehat{\bigotimes}_{\beta_k}^{k,s} X^{**}$ as a locally complemented subspace of $(\widehat{\bigotimes}_{\beta_k}^{k,s} X)^{**}$, implies that Θ_{β_k} is an isomorphic embedding. Specifically we obtain that, if X^{**} has the λ-AP, then, for all $v \in \widehat{\bigotimes}_{\beta_k}^{k,s} X^{**}$,

$$\lambda^{-k} \beta_k(v) \leqslant \|\Theta_{\beta_k}(v)\| \leqslant \beta_k(v).$$

Consequently, when X^{**} has the MAP we derive the following corollary:

Corollary 3.4. *Let β_k be an s-tensor norm of order k. If X^{**} has the MAP, then Θ_{β_k} is an isometry with its image.*

For the case of an injective s-tensor norm of order k, β_k, the thesis of Corollary 3.4 holds without the hypothesis of X^{**} having the approximation property.

Proposition 3.5. *Let β_k be an injective s-tensor norm of order k. Then,*

$$\Theta_{\beta_k} : \widehat{\bigotimes}_{\beta_k}^{k,s} X^{**} \to \left(\widehat{\bigotimes}_{\beta_k}^{k,s} X \right)^{**}$$

is an isometry with its image.

To prove it, we need first the following lemma.

Lemma 3.6. *Let β_k be an s-tensor norm of order k and let $T \in \mathcal{L}(X, Y)$. Then, the following diagram commutes*

$$
\begin{array}{ccc}
\widehat{\bigotimes}_{\beta_k}^{k,s} X^{**} & \xrightarrow{\Theta_{\beta_k}} & (\widehat{\bigotimes}_{\beta_k}^{k,s} X)^{**} \\
\downarrow{\scriptstyle \otimes^{k,s} T^{**}} & & \downarrow{\scriptstyle (\otimes^{k,s} T)^{**}} \\
\widehat{\bigotimes}_{\beta_k}^{k,s} Y^{**} & \xrightarrow{\Theta_{\beta_k}} & (\widehat{\bigotimes}_{\beta_k}^{k,s} Y)^{**}
\end{array}
\tag{5}
$$

Proof. It is enough to prove that the diagram commutes when applied to elementary tensors. So, we need to see that $\Theta_{\beta_k}(\otimes^{k,s} T^{**}(\otimes^k z)) = (\otimes^{k,s} T)^{**}(\Theta_{\beta_k}(\otimes^k z))$, for all $z \in X^{**}$. Let $P \in \mathcal{P}_{\beta_k}(^k Y) = (\widehat{\bigotimes}_{\beta_k}^{k,s} Y)^*$, then

$$\Theta_{\beta_k}(\otimes^{k,s} T^{**}(\otimes^k z))(P) = \Theta_{\beta_k}(\otimes^k T^{**} z)(P) = \langle \overline{P}, \otimes^k T^{**} z \rangle = \overline{P}(T^{**} z).$$

On the other hand,

$$\left(\otimes^{k,s} T\right)^{**}\left(\Theta_{\beta_k}\left(\otimes^k z\right)\right)(P) = \left\langle \Theta_{\beta_k}\left(\otimes^k z\right), \left(\otimes^k T\right)^*(P)\right\rangle = \left\langle \overline{\left(\otimes^k T\right)^*(P)}, \otimes^k z\right\rangle$$
$$= \left\langle \overline{P \circ \otimes^k T}, \otimes^k z\right\rangle = \overline{P}\left(T^{**} z\right). \qquad \square$$

Proof of Proposition 3.5. Consider the canonical isometric inclusions:

$$i_X : X \to C(B_{X^*}), \qquad i_X^{**} : X^{**} \to C(B_{X^*})^{**}.$$

Now, by Lemma 3.6, the diagram commutes:

$$
\begin{array}{ccc}
\widehat{\bigotimes}_{\beta_k}^{k,s} X^{**} & \xrightarrow{\;\Theta_{\beta_k}\;} & (\widehat{\bigotimes}_{\beta_k}^{k,s} X)^{**} \\[2mm]
\downarrow{\scriptstyle \otimes^{k,s} i_X^{**}} & & \downarrow{\scriptstyle (\otimes^{k,s} i_X)^{**}} \\[2mm]
\widehat{\bigotimes}_{\beta_k}^{k,s} C(B_{X^*})^{**} & \xrightarrow{\;\Theta_{\beta_k}\;} & (\widehat{\bigotimes}_{\beta_k}^{k,s} C(B_{X^*}))^{**}
\end{array}
$$

By Lemma 4.4, Corollary 1 of 23.2 and Corollary 1 of 21.6 in [19] we know that $C(B_{X^*})^{**}$ has the MAP. Thus, by Corollary 3.4, $\Theta_{\beta_k} : \widehat{\bigotimes}_{\beta_k}^{k,s} C(B_{X^*})^{**} \to (\widehat{\bigotimes}_{\beta_k}^{k,s} C(B_{X^*}))^{**}$ is an isometry. Since β_k is an injective s-tensor norm, the mappings $\otimes^{k,s} i_X^{**}$ and $(\otimes^{k,s} i_X)^{**}$ are also isometries. Finally, the commutativity of the diagram yields the isometry of the desired mapping $\Theta_{\beta_k} : \widehat{\bigotimes}_{\beta_k}^{k,s} X^{**} \to (\widehat{\bigotimes}_{\beta_k}^{k,s} X)^{**}$. $\quad\square$

Question 3.7. *If Θ_{β_k} is an isomorphic embedding, does it imply that Θ_{β_k} embeds $\widehat{\bigotimes}_{\beta_k}^{k,s} X^{**}$ as a locally complemented subspace of $(\widehat{\bigotimes}_{\beta_k}^{k,s} X)^{**}$?*

Remark 3.8. Another result for 2-fold tensor products of spaces without AP was proved in [10, Corollary 4] by Cabello and the third author. They showed that $\Theta_\pi : X^{**} \widehat{\bigotimes}_\pi X^{**} \to (X \widehat{\bigotimes}_\pi X)^{**}$ is an isomorphic embedding if X has type 2 and X^* has cotype 2. Examples of this result are $\mathcal{K}(\mathcal{H})$, $\mathcal{L}(\mathcal{H})$ and the Pisier's space [35] which have no uniformly complemented finite dimensional subspaces. Note that Pisier's space and $\mathcal{L}(\mathcal{H}) = \mathcal{K}(\mathcal{H})^{**}$ both fail the AP.

A canonical commutative diagram allows us to translate this statement to the symmetric case. Thus, we have: if X has type 2 and X^* has cotype 2, then $\Theta_{\pi_{2,s}} : \widehat{\bigotimes}_{\pi_{2,s}}^{2,s} X^{**} \to (\widehat{\bigotimes}_{\pi_{2,s}}^{2,s} X)^{**}$ is an isomorphic embedding.

Remark 3.9 *(The holomorphic case).* The previous results have some consequences in the holomorphic setting. To state them, we need to consider 'the same' s-tensor norm β_k of order k, for each k. It is clear what we mean by 'the same' when $\beta_k = \pi_{k,s}$ or $\beta_k = \varepsilon_{k,s}$, for all k. For the general case we refer to the concept of 'coherent sequence of polynomial ideals' defined in [14] or [13].

Hence, we say that $\beta = (\beta_k)_k$ is an s-*tensor norm* if, for each k, β_k is an s-tensor norm of order k and the sequence $\{\mathcal{P}_{\beta_k}\}_k$ is coherent. For this particular case of scalar-valued polynomial ideals dual to symmetric tensor norms, the notion of 'coherency' turns up to be the following:

The sequence $\{\mathcal{P}_{\beta_k}\}_k$ is coherent if there exist positive constants C and D such that for every Banach space X, the following conditions hold for every k:

(i) For each $P \in \mathcal{P}_{\beta_{k+1}}(^{k+1}X)$ and $a \in X$, $x \mapsto \overset{\vee}{P}(a, x, \ldots, x)$ belongs to $\mathcal{P}_{\beta_k}(^kX)$ and

$$\left\| x \mapsto \overset{\vee}{P}(a, x, \ldots, x) \right\|_{\mathcal{P}_{\beta_k}(^kX)} \leqslant C \|P\|_{\mathcal{P}_{\beta_{k+1}}(^{k+1}X)} \|a\|.$$

(ii) For each $P \in \mathcal{P}_{\beta_k}(^kX)$ and $\gamma \in X^*$, γP belongs to $\mathcal{P}_{\beta_{k+1}}(^{k+1}X)$ and

$$\|\gamma P\|_{\mathcal{P}_{\beta_{k+1}}(^{k+1}X)} \leqslant D \|\gamma\| \|P\|_{\mathcal{P}_{\beta_k}(^kX)}.$$

Following [13], to a given s-tensor norm β, we can associate a Fréchet space of holomorphic functions of bounded type:

$$H_{b\beta}(X) = \left\{ f \in H(X): \frac{d^k f(0)}{k!} \in \mathcal{P}_{\beta_k}(^kX) \text{ for all } k \text{ and } \limsup_{k \to \infty} \left\| \frac{d^k f(0)}{k!} \right\|_{\mathcal{P}_{\beta_k}(^kX)}^{\frac{1}{k}} = 0 \right\}.$$

Analogously, we can consider holomorphic functions of bounded type associated to β defined on an open ball $B_r(x)$ of center x and radius r:

$$H_{b\beta}(B_r(x))$$

$$= \left\{ f \in H(B_r(x)): \frac{d^k f(x)}{k!} \in \mathcal{P}_{\beta_k}(^kX) \text{ for all } k \text{ and } \limsup_{k \to \infty} \left\| \frac{d^k f(x)}{k!} \right\|_{\mathcal{P}_{\beta_k}(^kX)}^{\frac{1}{k}} \leqslant \frac{1}{r} \right\}.$$

Galindo, Maestre and Rueda in [26] introduced and developed the concept of 'R-Schauder decomposition'. For $0 < R \leqslant \infty$, a sequence of Banach spaces $(E_k, \|\cdot\|_k)$ is an *R-Schauder decomposition* of a Fréchet space E if it is a Schauder decomposition and verifies the condition: for every sequence $(x_k)_k$, with $x_k \in E_k$, the series $\sum_{k=1}^{\infty} x_k$ converges in E if and only if

$$\limsup_k \|x_k\|_k^{\frac{1}{k}} \leqslant \frac{1}{R}.$$

For an s-tensor norm β we know from [34, Propositions 3.2.11 and 3.2.53] the following:

- $\{\mathcal{P}_{\beta_k}(^kX)\}_k$ is an ∞-Schauder decomposition of $H_{b\beta}(X)$.
- $\{\mathcal{P}_{\beta_k}(^kX)\}_k$ is an r-Schauder decomposition of $H_{b\beta}(B_r(x))$.

Now, from Theorem 3.3 and Corollary 3.4, invoking some results of [26] and arguing as in [10, Theorem 4] we obtain, for any s-tensor norm β, the following:

- If X^{**} has the BAP, then $H_{b\beta}(X^{**})$ is a complemented subspace of $H_{b\beta}(X)^{**}$.
- If X^{**} has the MAP, then $H_{b\beta}(B_r^{**}(x))$ is a complemented subspace of $H_{b\beta}(B_r(x))^{**}$ (where $B_r^{**}(x)$ means the ball of X^{**} with center $x \in X$ and radius r).

4. Unique norm preserving extension for a polynomial belonging to an ideal

Godefroy gave in [27] a characterization of norm-one functionals having unique norm preserving extensions to the bidual as the points of S_{X^*} where the identity is w^*–w continuous (see also [28, Lemma III.2.14]).

Lemma 4.1. *Let X be a Banach space and $x^* \in S_{X^*}$. The following are equivalent*:

(i) x^* *has a unique norm preserving extension to a functional on X^{**}.*
(ii) *The function $Id_{B_{X^*}} : (B_{X^*}, w^*) \longrightarrow (B_{X^*}, w)$ is continuous at x^*.*

Aron, Boyd and Choi presented in [5] a polynomial version of this result:

Proposition 4.2. *Let X be a Banach space such that X^{**} has the MAP and let $P \in S_{\mathcal{P}(^k X)}$. The following are equivalent*:

(i) P *has a unique norm preserving extension to $\mathcal{P}(^k X^{**})$.*
(ii) *If $\{P_\alpha\}_\alpha \subset B_{\mathcal{P}(^k X)}$ converges pointwise to P, then $\{\overline{P_\alpha}\}_\alpha$ converges pointwise to $\overline{P}$ in X^{**}.*

We are interested on having a similar characterization for unique norm preserving extensions to the bidual of polynomials belonging to a maximal (scalar-valued) polynomial ideal. In this case, obviously, the norm that we want to preserve is the ideal norm.

Let β_k be an s-tensor norm of order k and let $P \in \mathcal{P}_{\beta_k}(^k X)$. Since we have already mentioned that the Aron–Berner extension preserves the ideal norm for maximal polynomial ideals [15], should P has unique norm preserving extension, this extension ought to be $\overline{P}$.

To prove our result we need the following equivalence between different topologies for the convergence of nets of polynomials in an ideal unit ball. The proof is straightforward.

Lemma 4.3. *Suppose that the polynomial P and the net $\{P_\alpha\}_\alpha$ are contained in the unit ball of $\mathcal{P}_{\beta_k}(^k X)$, where β_k is an s-tensor norm of order k. Then, the following are equivalent*:

(i) $P_\alpha(x) \to P(x)$ *for all $x \in X$.*
(ii) $P_\alpha \to P$ *for the topology $\sigma(\mathcal{P}_{\beta_k}(^k X), \widehat{\bigotimes}_{\beta_k}^{k,s} X)$.*
(iii) $P_\alpha \to P$ *for the topology $\sigma(\mathcal{P}(^k X), \widehat{\bigotimes}_{\pi_{k,s}}^{k,s} X)$.*

Theorem 4.4. *Let β_k be an s-tensor norm of order k and suppose that X^{**} has the MAP. Consider a polynomial $P \in \mathcal{P}_{\beta_k}(^k X)$ with $\|P\|_{\mathcal{P}_{\beta_k}(^k X)} = 1$. Then, the following are equivalent*:

(i) P *has a unique norm preserving extension to $\mathcal{P}_{\beta_k}(^k X^{**})$.*
(ii) *The β_k-Aron–Berner extension $(AB)_{\beta_k} : (B_{\mathcal{P}_{\beta_k}(^k X)}, w^*) \longrightarrow (B_{\mathcal{P}_{\beta_k}(^k X^{**})}, w^*)$ is continuous at P.*
(iii) *If the net $\{P_\alpha\}_\alpha \subset B_{\mathcal{P}_{\beta_k}(^k X)}$ converges pointwise to P, then $\{\overline{P_\alpha}\}_\alpha$ converges pointwise to $\overline{P}$ in X^{**}.*

Proof. (i) $\Rightarrow$ (ii) Let $\{P_\alpha\}_\alpha \subset B_{\mathcal{P}_{\beta_k}(^kX)}$ such that $P_\alpha \overset{w^*}{\to} P$. We want to see that $\overline{P}_\alpha \overset{w^*}{\to} \overline{P}$ in $\mathcal{P}_{\beta_k}(^kX^{**})$. By the compactness of $(B_{\mathcal{P}_{\beta_k}(^kX^{**})}, w^*)$, the net $\{\overline{P}_\alpha\}_\alpha$ has a subnet $\{\overline{P}_\gamma\}_\gamma$ w^*-convergent to a polynomial $Q \in B_{\mathcal{P}_{\beta_k}(^kX^{**})}$.

For each $x \in X$, we have, on one hand, that $\overline{P}_\gamma(x) = P_\gamma(x) \to P(x)$ and, on the other hand, that $\overline{P}_\gamma(x) \to Q(x)$. So, $Q|_X = P$. Also, $\|Q\| \leqslant 1 = \|P\|$ implies $\|Q\|_{\mathcal{P}_{\beta_k}(^kX^{**})} = \|P\|_{\mathcal{P}_{\beta_k}(^kX)}$. This means that Q is a norm preserving extension of P and by (i) it should be $Q = \overline{P}$. Since for every subnet of $\{P_\alpha\}_\alpha$ we can find a sub-subnet such that the Aron–Berner extensions are w^*-convergent to $\overline{P}$, we conclude that $\overline{P}_\alpha \overset{w^*}{\to} \overline{P}$.

(ii) $\Rightarrow$ (i) Let $Q \in \mathcal{P}_{\beta_k}(^kX^{**})$ be an extension of P with $\|Q\|_{\mathcal{P}_{\beta_k}(^kX^{**})} = 1$. From Corollary 3.4, $\Theta_{\beta_k} : \widehat{\bigotimes}^{k,s}_{\beta_k} X^{**} \longrightarrow (\widehat{\bigotimes}^{k,s}_{\beta_k} X)^{**}$ is an isometry. Due to this each polynomial $Q \in \mathcal{P}_{\beta_k}(^kX^{**}) = (\widehat{\bigotimes}^{k,s}_{\beta_k} X^{**})^*$ has a Hahn–Banach extension $\widetilde{Q} \in (\widehat{\bigotimes}^{k,s}_{\beta_k} X)^{***} = \mathcal{P}_{\beta_k}(^kX)^{**}$. By Goldstine's Theorem, there exist a net $\{P_\alpha\}_\alpha \subset B_{\mathcal{P}_{\beta_k}(^kX)}$ such that $P_\alpha \overset{w^*}{\to} \widetilde{Q}$, where w^* means here the topology $\sigma(\mathcal{P}_{\beta_k}(^kX)^{**}, \mathcal{P}_{\beta_k}(^kX)^*)$.

Let $u \in \widehat{\bigotimes}^{k,s}_{\beta_k} X \subset \mathcal{P}_{\beta_k}(^kX)^*$. So we have

$$\langle P_\alpha, u \rangle \to \langle \widetilde{Q}, u \rangle = \langle Q, u \rangle = \langle P, u \rangle.$$

This means that $P_\alpha \overset{w^*}{\to} P$, where w^* denotes here the topology $\sigma(\mathcal{P}_{\beta_k}(^kX), \widehat{\bigotimes}^{k,s}_{\beta_k} X)$. By (ii), this implies that $\overline{P}_\alpha \to \overline{P}$ for the topology $\sigma(\mathcal{P}_{\beta_k}(^kX^{**}), \widehat{\bigotimes}^{k,s}_{\beta_k} X^{**})$.

Now, if $v \in \widehat{\bigotimes}^{k,s}_{\beta_k} X^{**}$, it follows that

$$\langle \overline{P}_\alpha, v \rangle \to \langle \overline{P}, v \rangle.$$

But also, since $v \in \mathcal{P}_{\beta_k}(^kX)^*$,

$$\langle \overline{P}_\alpha, v \rangle = \langle v, P_\alpha \rangle \to \langle v, \widetilde{Q} \rangle = \langle Q, v \rangle.$$

Therefore, $\overline{P} = Q$.

The equivalence between (ii) and (iii) is a consequence of the previous lemma. $\quad\square$

For the particular case of β_k being an injective s-tensor norm, the same argument but applying Proposition 3.5 instead of Corollary 3.4, yields to a version of Theorem 4.4 without the hypothesis of metric approximation property.

Corollary 4.5. *Let β_k be an injective s-tensor norm of order k. Consider a polynomial $P \in \mathcal{P}_{\beta_k}(^kX)$ with $\|P\|_{\mathcal{P}_{\beta_k}(^kX)} = 1$. Then, the following are equivalent:*

(i) *P has a unique norm preserving extension to $\mathcal{P}_{\beta_k}(^kX^{**})$.*
(ii) *The "β_k"-Aron–Berner extension $(AB)_{\beta_k} : (B_{\mathcal{P}_{\beta_k}(^kX)}, w^*) \longrightarrow (B_{\mathcal{P}_{\beta_k}(^kX^{**})}, w^*)$ is continuous at P.*
(iii) *If the net $\{P_\alpha\}_\alpha \subset B_{\mathcal{P}_{\beta_k}(^kX)}$ converges pointwise to P, then $\{\overline{P}_\alpha\}_\alpha$ converges pointwise to $\overline{P}$ in X^{**}.*

5. Q_β-reflexivity

In [6] Aron and Dineen considered the problem of obtaining a polynomial functional representation of the bidual of $\mathcal{P}(^k X)$. More precisely, they asked when the space $\mathcal{P}(^k X)^{**}$ is isomorphic to $\mathcal{P}(^k X^{**})$ in a canonical way. Spaces with this property are called Q-reflexive. A reflexive Banach space X with the approximation property is Q-reflexive if and only if $\mathcal{P}(^k X)$ is reflexive, for all k.

In an analogous way, we consider a sort of Q-reflexivity for a maximal polynomial ideal. Recall that a maximal polynomial ideal is the dual of the symmetric tensor product endowed with a finitely generated s-tensor norm (see Remark 3.9 for the definition of an s-tensor norm $\beta = (\beta_k)_k$).

Definition 5.1. Let β be an s-tensor norm. We say that a Banach space X is Q_β^k-reflexive if $\Theta_{\beta_k}^* : \mathcal{P}_{\beta_k}(^k X)^{**} \longrightarrow \mathcal{P}_{\beta_k}(^k X^{**})$ is an isomorphism. We say that X is Q_β-reflexive if it is Q_β^k-reflexive for all k.

Naturally, Q_{π_s}-reflexive spaces are the usual Q-reflexive spaces. It follows from [38, Proposition 2.2] that if X^{**} has the AP and the Radon–Nikodým property (RNP), then X is $Q_{\pi_s}^k$-reflexive if and only if $\mathcal{P}(^k X) = \mathcal{P}_w(^k X)$ (for the multilinear case see [10, Corollary 3]). We obtain here a kind of "predual version" of that result, a condition for the $Q_{\varepsilon_s}^k$-reflexivity.

Theorem 5.2. *Suppose that X^{**} has the AP and X^* has the RNP. Then, X is $Q_{\varepsilon_s}^k$-reflexive if and only if $\mathcal{P}(^k X^*) = \mathcal{P}_w(^k X^*)$.*

Proof. First we consider the Borel transformation $B_k : \mathcal{P}(^k X^*)^* \to \mathcal{P}_I(^k X^{**})$ on X^*. B_k is the adjoint of the natural operator $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X^{**} \to (\widehat{\bigotimes}_{\pi_{k,s}}^{k,s} X^*)^* = \mathcal{P}(^k X^*)$ (which is always an isometric embedding).

Now, the RNP of X^* implies that $\mathcal{P}_I(^k X) = \mathcal{P}_N(^k X)$ [9,12], and the AP of X^* yields the equality $\mathcal{P}_N(^k X) = \widehat{\bigotimes}_{\pi_{k,s}}^{k,s} X^*$. Then,

$$\mathcal{P}_I(^k X)^{**} = \mathcal{P}_N(^k X)^{**} = \left(\widehat{\bigotimes}_{\pi_{k,s}}^{k,s} X^*\right)^{**} = \mathcal{P}(^k X^*)^*.$$

It follows that $\mathcal{P}_I(^k X)^{**}$ is isomorphic to $\mathcal{P}_I(^k X^{**})$ through $\Theta_{\varepsilon_{k,s}}^*$ if and only if B_k is an isomorphism.

On the other hand, if X^{**} has the approximation property then every polynomial in $\mathcal{P}_w(^k X^*)$ is uniformly approximable on B_{X^*} by finite type polynomials and hence $\mathcal{P}_w(^k X^*)$ is isometrically isomorphic to $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X^{**}$ [7]. Hence,

$$\mathcal{P}_w(^k X^*)^* = \left(\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X^{**}\right)^* = \mathcal{P}_I(^k X^{**}).$$

Therefore, B_k is an isomorphism if and only if $\mathcal{P}(^k X^*) = \mathcal{P}_w(^k X^*)$. $\square$

In the above result, the RNP on X^* can be replaced by weaker hypothesis that ℓ_1 is not a subspace of $\widehat{\bigotimes}_{\varepsilon_{k,s}}^{k,s} X$ (see [9,12]).

Remark 5.3. (a) If $\mathcal{P}_I({}^k X^{**}) = \widehat{\bigotimes}_{\pi_{k,s}}^{k,s} X^{***}$ (for example if X^{***} has the AP and the RNP), then X is $Q_{\varepsilon_s}^k$-reflexive if and only if X^* is $Q_{\pi_s}^k$-reflexive. Indeed, if X^* is $Q_{\pi_s}^k$-reflexive, one has

$$\mathcal{P}_I({}^k X^{**}) = \widehat{\bigotimes}_{\pi_{k,s}}^{k,s} X^{***} \simeq \left(\widehat{\bigotimes}_{\pi_{k,s}}^{k,s} X^*\right)^{**} = \mathcal{P}_I({}^k X)^{**}.$$

On the other hand, X is a $Q_{\varepsilon_s}^k$-reflexive space, we have

$$\widehat{\bigotimes}_{\pi_{k,s}}^{k,s} X^{***} = \mathcal{P}_I({}^k X^{**}) \simeq \mathcal{P}_I({}^k X)^{**} = \left(\widehat{\bigotimes}_{\pi_{k,s}}^{k,s} X^*\right)^{**}.$$

Note that in both cases we use that whenever the equality $\mathcal{P}_I({}^k X^{**}) = \widehat{\bigotimes}_{\pi_{k,s}}^{k,s} X^{***}$ holds, it is also valid that $\mathcal{P}_I({}^k X) = \widehat{\bigotimes}_{\pi_{k,s}}^{k,s} X^*$.

(b) If, for every $k \geqslant 1$, each k-linear form on X is weakly continuous on bounded sets (that is, $\mathcal{L}({}^k X) = \mathcal{L}_w({}^k X)$) and all the spaces $\mathcal{L}({}^k X)$ have the approximation property then

$$\mathcal{P}\left({}^k \widehat{\bigotimes}_{\pi_{n,s}}^{n,s} X\right) = \mathcal{P}_w\left({}^k \widehat{\bigotimes}_{\pi_{n,s}}^{n,s} X\right),$$

for all $k, n \geqslant 1$.

Indeed, from the hypothesis we obtain the following equalities for the nonsymmetric case:

$$\mathcal{L}\left({}^k \widehat{\bigotimes}_{\pi_n}^{n} X\right) = \left[\widehat{\bigotimes}_{\pi_k}^{k}\left(\widehat{\bigotimes}_{\pi_n}^{n} X\right)\right]^* = \left(\widehat{\bigotimes}_{\pi_{nk}}^{nk} X\right)^* = \mathcal{L}({}^{nk} X) = \mathcal{L}_w({}^{nk} X)$$

$$= \widehat{\bigotimes}_{\varepsilon_{nk}}^{nk} X^* = \widehat{\bigotimes}_{\varepsilon_k}^{k}\left(\widehat{\bigotimes}_{\varepsilon_n}^{n} X^*\right) = \widehat{\bigotimes}_{\varepsilon_k}^{k}\left(\mathcal{L}_w({}^n X)\right)$$

$$= \widehat{\bigotimes}_{\varepsilon_k}^{k}\left(\mathcal{L}({}^n X)\right) = \widehat{\bigotimes}_{\varepsilon_k}^{k}\left(\left(\widehat{\bigotimes}_{\pi_n}^{n} X\right)^*\right) = \mathcal{L}_w\left({}^k \widehat{\bigotimes}_{\pi_n}^{n} X\right).$$

Thus, as a consequence, every k-homogeneous polynomial on $\widehat{\bigotimes}_{\pi_n}^{n} X$ is weakly continuous on bounded sets and since $\widehat{\bigotimes}_{\pi_{n,s}}^{n,s} X$ is isomorphic to a complemented subspace of $\widehat{\bigotimes}_{\pi_n}^{n} X$ the same happens for polynomials in the symmetric tensor product.

Example 5.4.

(i) The Tsirelson space T and the predual of the Tsirelson–James space T_J^* (which we denote T_J) are Q_{ε_s}-reflexive (recall that $\mathcal{L}({}^k T^*) = \mathcal{L}_w({}^k T^*)$ and $\mathcal{L}({}^k T_J^*) = \mathcal{L}_w({}^k T_J^*)$ [1,6]). Moreover, for all $n > 1$, the spaces $X = \widehat{\bigotimes}_{\varepsilon_{n,s}}^{n,s} T$ and $Y = \widehat{\bigotimes}_{\varepsilon_{n,s}}^{n,s} T_J$ are Q_{ε_s}-reflexive. Since $X^* = \widehat{\bigotimes}_{\pi_{n,s}}^{n,s} T^*$ and $Y^* = \widehat{\bigotimes}_{\pi_{n,s}}^{n,s} T_J^*$ are separable dual spaces they have RNP. Also, $X^{**} = \mathcal{P}({}^n T^*)$ and $Y^{**} = \mathcal{P}({}^n T_J^*)$ have the approximation property. Then, the Q_{ε_s}-reflexivity of X and Y follows from Theorem 5.2 and Remark 5.3(b). Note that Y is not quasi-reflexive (i.e., Y^{**}/Y is infinite dimensional).

(ii) The space ℓ_p is $Q_{\varepsilon_s}^k$-reflexive if and only if $k < p^*$ (where p^* is the conjugate of p). The space L_p ($1 < p < \infty$) contains a complemented copy of ℓ_2. Thus, L_p is not $Q_{\varepsilon_s}^k$-reflexive for any $k > 1$.

As seen in the proof of Theorem 5.2, we can translate the statement of that theorem in the following way: if X^{**} has the AP and X^* has the RNP, then X is $Q^k_{\varepsilon_s}$-reflexive if and only if $\mathcal{P}(^kX^*) = \widehat{\bigotimes}^{k,s}_{\varepsilon_{k,s}} X^{**}$. We now present a similar result for the case of the symmetric tensor norm $\beta = /\pi_s\backslash$. Recall that $(\widehat{\bigotimes}^{k,s}_{/\pi_{k,s}\backslash} X)^* = \mathcal{P}_e(^kX)$ is the ideal of extendible k-homogeneous polynomials.

Theorem 5.5. *Let X be a Banach space such that X^* has the BAP and the RNP. Then, X is $Q^k_{/\pi_s\backslash}$-reflexive if and only if $\mathcal{P}_{\backslash\varepsilon_{k,s}/}(^kX^*) = \widehat{\bigotimes}^{k,s}_{/\pi_{k,s}\backslash} X^{**}$.*

Proof. Since X is Asplund and X^* has the BAP then, by the comments after [17, Corollary 2.4], we have $\mathcal{P}_e(^kX) = \widehat{\bigotimes}^{k,s}_{\backslash\varepsilon_{k,s}/} X^*$. More precisely, the mapping

$$J_{\backslash\varepsilon_{k,s}/} : \widehat{\bigotimes}^{k,s}_{\backslash\varepsilon_{k,s}/} X^* \to \mathcal{P}_e(^kX),$$

given by $J_{\backslash\varepsilon_{k,s}/}(\otimes^k \phi) = \phi^k$ is an isometric isomorphism. Thus, $\mathcal{P}_e(^kX)^* = (\widehat{\bigotimes}^{k,s}_{\backslash\varepsilon_{k,s}/} X^*)^* = \mathcal{P}_{\backslash\varepsilon_{k,s}/}(^kX^*)$ via $(J_{\backslash\varepsilon_{k,s}/})^*$.

Consider the following two mappings

$$\Theta_{/\pi_{k,s}\backslash} : \widehat{\bigotimes}^{k,s}_{/\pi_{k,s}\backslash} X^{**} \to \mathcal{P}_e(^kX)^* \quad \text{and}$$

$$J_{/\pi_{k,s}\backslash} : \widehat{\bigotimes}^{k,s}_{/\pi_{k,s}\backslash} X^{**} \to \left(\widehat{\bigotimes}^{k,s}_{\backslash\varepsilon_{k,s}/} X^*\right)^* = \mathcal{P}_{\backslash\varepsilon_{k,s}/}(^kX^*),$$

where $\Theta_{/\pi_{k,s}\backslash}(\otimes^k z)(P) = \overline{P}(z)$ and $J_{/\pi_{k,s}\backslash}(\otimes^k z) = z^k$.

Notice that, with the above identifications, these two mappings are equal. Indeed, since $J_{\backslash\varepsilon_{k,s}/}(\bigotimes^{k,s} X^*)$ is dense in $\mathcal{P}_e(^kX)$, by linearity we only have to check the following equality

$$\Theta_{/\pi_{k,s}\backslash}(\otimes^k z)(J_{\backslash\varepsilon_{k,s}/}(\otimes^k \phi))$$
$$= \Theta_{/\pi_{k,s}\backslash}(\otimes^k z)(\phi^k) = \overline{\phi^k}(z) = (z(\phi))^k = z^k(\phi) = J_{/\pi_{k,s}\backslash}(\otimes^k z)(\phi).$$

In consequence, we obtain that X is $Q^k_{/\pi_s\backslash}$-reflexive if and only if $J_{/\pi_{k,s}\backslash}$ is an isomorphism. And this is equivalent to the identity $\mathcal{P}_{\backslash\varepsilon_{k,s}/}(^kX^*) = \widehat{\bigotimes}^{k,s}_{/\pi_{k,s}\backslash} X^{**}$. $\square$

Carando and the second author studied in [17] the symmetric Radon–Nikodým property: A finitely generated s-tensor norm of order k, β_k, has the symmetric Radon–Nikodým property if the canonical mapping $\widehat{\bigotimes}^{k,s}_{\beta_k} \ell_1 \twoheadrightarrow (\widehat{\bigotimes}^{k,s}_{\beta_k^*} c_0)^*$ is an isometric isomorphism. For adjoints of projective tensor norms having this property the previous result is canonically extended:

Theorem 5.6. *Let β_k be a projective s-tensor norm of order k with the symmetric Radon–Nikodým property and let X be an Asplund space such that X^* has the BAP. Then, X is $Q^k_{\beta^*}$-reflexive if and only if $\mathcal{P}_{\beta_k}(^kX^*) = \widehat{\bigotimes}^{k,s}_{\beta_k^*} X^{**}$.*

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References

[1] A. Alencar, K. Floret, Weak continuity of multilinear mappings on Tsirelson's space, Quaest. Math. 21 (1998) 177–186.

[2] E.M. Alfsen, E.G. Effros, Structure in real Banach spaces I, II, Ann. of Math. 96 (1972) 98–173.

[3] J.M. Ansemil, K. Floret, The symmetric tensor product of a direct sum of locally convex spaces, Studia Math. 129 (1998) 285–295.

[4] R. Aron, P. Berner, A Hahn–Banach extensions theorem for analytic mappings, Bull. Soc. Math. France 106 (1978) 3–24.

[5] R. Aron, C. Boyd, Y.S. Choi, Unique Hahn–Banach theorems for spaces homogeneous polynomials, J. Aust. Math. Soc. 70 (2001) 387–400.

[6] R.M. Aron, S. Dineen, Q-reflexive Banach spaces, Rocky Mountain J. Math. 27 (4) (1997) 1009–1025.

[7] R.M. Aron, J.B. Prolla, Polynomial approximation of differentiable functions on Banach spaces, J. Reine Angew. Math. 313 (1980) 195–216.

[8] C. Boyd, S. Lassalle, Extreme and exposed points of spaces of integral polynomials, Proc. Amer. Math. Soc. 138 (2010) 1415–1420.

[9] C. Boyd, R. Ryan, Geometric theory of spaces of integral polynomials and symmetric tensor products, J. Funct. Anal. 179 (2001) 18–42.

[10] F. Cabello Sánchez, R. García, The bidual of tensor product of Banach spaces, Rev. Mat. Iberoam. 21 (2005) 843–861.

[11] F. Cabello Sánchez, D. Pérez-García, I. Villanueva, Unexpected subspaces of tensor products, J. Lond. Math. Soc. 74 (2) (2006) 512–526.

[12] D. Carando, V. Dimant, Duality in spaces of nuclear and integral polynomials, J. Math. Anal. Appl. 241 (2000) 107–121.

[13] D. Carando, V. Dimant, S. Muro, Hypercyclic convolution operators on Fréchet spaces of analytic functions, J. Math. Anal. Appl. 336 (2) (2007) 1324–1340.

[14] D. Carando, V. Dimant, S. Muro, Coherent sequences of polynomial ideals on Banach spaces, Math. Nachr. 282 (8) (2009) 1111–1133.

[15] D. Carando, D. Galicer, Extending polynomials in maximal and minimal ideals, Publ. Res. Inst. Math. Sci. 46 (3) (2010) 669–680.

[16] D. Carando, D. Galicer, The five basic lemmas of the metric theory of symmmetric tensor products, preprint.

[17] D. Carando, D. Galicer, The symmetric Radon–Nikodým property for tensor norms, J. Math. Anal. Appl. 375 (2011) 553–565.

[18] A.M. Davie, T.W. Gamelin, A theorem on polynomial-star approximation, Proc. Amer. Math. Soc. 106 (2) (1989) 351–356.

[19] A. Defant, K. Floret, Tensor norms and operator ideals, in: North-Holland Math. Stud., vol. 176, Amsterdam, 1993.

[20] V. Dimant, M-ideals of homogeneous polynomials, Studia Math. 202 (1) (2011) 81–104.

[21] S. Dineen, Complex Analysis on Infinite Dimensional Spaces, Monogr. Math., Springer-Verlag, London, 1999.

[22] S. Dineen, Extreme integral polynomials on a complex Banach space, Math. Scand. 92 (2003) 129–140.

[23] K. Floret, Natural norms on symmetric tensor products of normed spaces, in: Proceedings of the Second International Workshop on Functional Analysis, Trier, 1997, Note Mat. 17 (1999) 153–188.

[24] K. Floret, On ideals of n-homogeneous polynomials on Banach spaces, in: Topological Algebras with Applications to Differential Geometry and Mathematical Physics, Athens, 1999, Univ. Athens, Athens, 2002, pp. 19–38.

[25] K. Floret, S. Hunfeld, Ultrastability of ideals of homogeneous polynomials and multilinear mappings on Banach spaces, Proc. Amer. Math. Soc. 130 (5) (2002) 1425–1435.

[26] P. Galindo, M. Maestre, P. Rueda, Biduality in spaces of holomorphic functions, Math. Scand. 86 (1) (2000) 5–16.
[27] G. Godefroy, Points de Namioka, espaces normante, applications á la Théorie isométrique de la dualité, Israel J. Math. 38 (1981) 209–220.
[28] P. Harmand, D. Werner, W. Werner, M-ideal in Banach Spaces and Banach Algebras, Lecture Notes in Math., vol. 1547, Springer, Berlin, 1993.
[29] J.A. Jaramillo, A. Prieto, I. Zalduendo, The bidual of the space of polynomials on a Banach space, Math. Proc. Cambridge Philos. Soc. 122 (1997) 457–471.
[30] N.J. Kalton, Locally complemented subspaces and $\mathcal{L}_p$-spaces for $0 < p < 1$, Math. Nachr. 115 (1984) 71–97.
[31] S. Lassalle, I. Zalduendo, To what extent does the dual Banach space E' determine the polynomials over E?, Ark. Mat. 38 (2000) 343–354.
[32] V. Lima, Ideal-structure in tensor products, preprint.
[33] J. Lindenstrauss, H.P. Rosenthal, The $\mathcal{L}_p$-spaces, Israel J. Math. 7 (1969) 325–349.
[34] S. Muro, Funciones holomorfas de tipo acotado e ideales de polinomios homogéneos en espacios de Banach, PhD thesis, Univ. de Buenos Aires, 2010.
[35] G. Pisier, Counterexamples to a conjecture of Grothendieck, Acta Math. 151 (1983) 181–208.
[36] W.M. Ruess, C.P. Stegall, Extreme points in the duals of operator spaces, Math. Ann. 261 (1982) 535–546.
[37] R.A. Ryan, B. Turett, Geometry of spaces of polynomials, J. Math. Anal. Appl. 221 (1998) 698–711.
[38] M. Venkova, Properties of Q-reflexive Banach spaces, J. Math. Anal. Appl. 264 (2001) 96–106.
[39] D. Werner, M-structure in tensor products of Banach spaces, Math. Scand. 61 (1) (1987) 149–164.
[40] W. Werner, Some results concerning the M-structure of operator spaces, Math. Ann. 282 (4) (1988) 545–553.
[41] I. Zalduendo, A canonical extension for analytic functions on Banach spaces, Trans. Amer. Math. Soc. 320 (1990) 747–763.
[42] I. Zalduendo, Extending polynomials on Banach spaces — a survey, Rev. Un. Mat. Argentina 46 (2) (2006) 45–72.