MIXED BOHR RADIUS IN SEVERAL VARIABLES

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ABSTRACT. Let $K(B_{\ell_p^n}, B_{\ell_q^n})$ be the n-dimensional (p,q)-Bohr radius for holomorphic functions on $\mathbb{C}^n$. That is, $K(B_{\ell_p^n}, B_{\ell_q^n})$ denotes the greatest number $r \geq 0$ such that for every entire function $f(z) = \sum_\alpha a_\alpha z^\alpha$ in n-complex variables, we have the following (mixed) Bohr-type inequality

$$\sup_{z \in r \cdot B_{\ell_q^n}} \sum_\alpha |a_\alpha z^\alpha| \leq \sup_{z \in B_{\ell_p^n}} |f(z)|,$$

where $B_{\ell_r^n}$ denotes the closed unit ball of the n-dimensional sequence space ℓ_r^n.

For every $1 \leq p, q \leq \infty$, we exhibit the exact asymptotic growth of the (p,q)-Bohr radius as n (the number of variables) goes to infinity.

1. INTRODUCTION

At the early twentieth century, during the course of his investigations on the famous Riemann ζ function, Harald Bohr [Boh13, Boh14] devoted great efforts in the study and development of a general theory of Dirichlet series. A Dirichlet series is just an expression of the form

$$D(s) = \sum_{n \geq 1} \frac{a_n}{n^s},$$

where $a_n \in \mathbb{C}$ and $s = \sigma + it$ is a complex variable. The regions of convergence, absolute convergence and uniform convergence of these series define half-planes of the form $[Re(s) > \sigma_0]$ in the complex field. Bohr was mainly interested in controlling the region of convergence of a series. To achieve this, he related different types of convergence and focused on finding the width of the greatest strip for which a Dirichlet series can converge uniformly but not absolutely. This question is popular and known nowadays as the *Bohr's absolute convergence problem*.

Although the solution of this problem problem appeared two decades after it was proposed (given by Bohnenblust and Hille [BH31, DSP+14] who showed that the maximum width of this

strip is $\frac{1}{2}$), Bohr made major contributions in the area in order to tackle it (arguably, even more important than the solution of the problem itself). He discovered a deep connection among Dirichlet series and power series in infinitely many variables. Given a Dirichlet series $D(s) = \sum_{n \geq 1} \frac{a_n}{n^s}$, he considered for each $n \in \mathbb{N}$ the prime decomposition $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ (where $(p_k)_{k \in \mathbb{N}}$ denote the sequence formed by the ordered primes) and defined $z = (p_1^{-s}, \ldots, p_r^{-s})$. Thus,

$$D(s) = \sum_{n \geq 1} a_n (p_1^{-s})^{\alpha_1} \ldots (p_r^{-s})^{\alpha_r} = \sum a_n z_1^{\alpha_1} \ldots z_r^{\alpha_r}.$$

This correspondence, known as the *Bohr transform* is not just formal: it gives an isometry between suitable spaces of Dirichlet series and power series [HLS97]. The Bohr transform allows to translate problems about Dirichlet series in terms of power series and tackle them with complex analysis techniques. This cycle of ideas brought Bohr to ask whether is possible to compare the absolute value of a power series in one complex variable with the sum of the absolute value of its coefficients. He manged to prove the following result nowadays referred as *Bohr's inequality*:

The radius $r = \frac{1}{3}$ is the largest value for which the following inequality holds:

$$\sum_{n \geq 0} |a_n| r^n \leq \sup_{z \in \mathbb{D}} |\sum_{n \geq 0} a_n z^n|, \tag{1}$$

for every entire function $f(z) = \sum_{n \geq 0} a_n z^n$ on the unit disk $\mathbb{D}$ such that $\sup_{z \in \mathbb{D}} |f(z)| < \infty$.

As a matter of fact, Bohr's paper [Boh14], compiled by G. H. Hardy from correspondence, indicates that Bohr initially obtained the radius $\frac{1}{6}$, but this was quickly improved to the sharp result by M. Riesz, I. Schur, and N. Wiener, independently. Bohr's article presents both his own proof and the one of his colleagues.

This interesting inequality was overlooked during many years until the end of the twentieth century. In particular, Dineen and Timoney [DT89], Dixon [Dix95], Boas and Khavinson [KB97], Aizenberg [Aiz00], Boas [Boa00] and Bombieri and Bourgain [BB04] retook this work and use it in different contexts and/or generalize it. Several of these authors analyzed if a similar phenomenon occurs for power series in many variables. For each Reinhardt domain $\mathcal{R}$, they introduced the notion of the *Bohr radius* $K(\mathcal{R})$ as the biggest $r \geq 0$ such that for every analytic function $f(z) = \sum_{\alpha} a_{\alpha} z^{\alpha}$ bounded on $\mathcal{R}$, it holds:

$$\sup_{z \in r \cdot \mathcal{R}} \sum_{\alpha} |a_{\alpha} z^{\alpha}| \leq \sup_{z \in \mathcal{R}} |f(z)|. \tag{2}$$

Note that with this notation, Bohr's inequality can be formulated simply as $K(\mathbb{D}) = \frac{1}{3}$. Surprisingly, the exact value of the Bohr radius is unknown for any other domain. The central results of [KB97, Boa00] contained a (partial) successful estimate for the Bohr radius for the complex unit balls of ℓ_p^n, $1 \leq p \leq \infty$.

The gap between the upper and lower estimates in this papers leaded many efforts to compute the exact asymptotic order of $K(B_{\ell_p^n})$, for $1 \leq p \leq \infty$.

To obtain the upper bounds Boas [Boa00] generalized in a very ingenious way a theorem of Kahane-Salem-Zygmund on random trigonometric polynomials [Kah93, Theorem 4 in Chapter 6], which gives (by the use of a probabilistic argument) the existence of homogeneous polynomials with "large coefficients" and uniform norm "relatively small". This technique (and some refinements of it, for example [DGM04, Bay12]) do the work when dealing with upper bounds.

The lower bound is a horse of a different color. In [DGM03] Defant, García and Maestre related the Bohr radius with some non-elementary concepts of the local theory of Banach spaces: unconditionality in spaces of homogeneous polynomials via some Banach-Mazur distance estimates (see also [DP06]). Although at that moment this did not give optimal asymptotic bounds, it started a way with which $K(B_{\ell_p^n})$ would be obtained.

They were Defant, Frerick, Ortega-Cerdá, Ounaïes and Seip [DFOC+11] who made an incredible contribution in the problem and managed to exhibit the exact asymptotic value of $K(B_{\ell_\infty^n})$. The authors involved again into the game the classical Bohnenblust-Hille inequality, which was used to compute Bohr's convergence width eighty years before. This inequality asserts that the $\ell_{\frac{2m}{m+1}}$-norm of the coefficients of a given m-homogeneous polynomial in n-complex variables is bounded by a constant *independent of n* times its supreme norm on the polydisk. Precisely, given $m \in \mathbb{N}$. there is a constant $C_m > 0$ such that for every m-homogeneous polynomial $P(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ in $\mathbb{C}^n$,

$$(3) \qquad \left(\sum_{|\alpha|=m} |a_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C_m \sup_{z \in \mathbb{D}^n} |P(z)|.$$

The groundbreaking progress consisted in showing that C_m is in fact hypercontractive; that is, C_m can be taken less than or equal to C^m for some absolute constant $C > 0$. With this at hand they proved that $K(B_{\ell_\infty^n})$ behaves asymptotically as $\sqrt{\frac{\log(n)}{n}}$ (other cornerstone of the paper is that they have also described the Sidon constant for the set of frequencies $\{\log(n) : n$ a positive integer $\leq N\}$). This paper, arguably in some sense, marked the path of the whole area over the last years. In fact much more can be said about $K(B_{\ell_\infty^n})$: Bayart, Pellegrino and

Seoane [BPSS14] managed to push these techniques further in an amazingly ingenious way to obtain that $\lim_{n\to\infty} \frac{K(B_{\ell_\infty^n})}{\sqrt{\frac{\log(n)}{n}}} = 1$.

Since $K(B_{\ell_\infty^n})$ bounds from below the radius $K(\mathcal{R})$ for any other Reinhardt domain $\mathcal{R}$, the range where $p \geq 2$ easily follows. The solution of the case $p < 2$, required quite different methods. A celebrated theorem proved independently by Pisier [Pis86] and Schütt [Sch78] allows to study unconditional bases in spaces of multilinear forms in terms of some invariants such as the local unconditional structure or the Gordon-Lewis property. These results have their counterpart in the context of spaces of polynomials as shown in [DDGM01], replacing the full tensor product by the symmetric one.

Defant and Frerick [DF11] (continuing their previous work given in [DF06]) established some sort of extension of Pisier-Schütt result to the symmetric tensor product with accurate bounds and gave a new estimate on the Gordon-Lewis constant of the symmetric tensor product. As a byproduct, they found the exact asymptotic growth for the Bohr radius on the unit ball of the spaces ℓ_p^n.

The aforementioned results give the following relation for the Bohr radius.

Theorem 1.1. [DFOC$^+$11, DF11] *For $1 \leq p \leq \infty$, we have*

$$(4) \qquad\qquad K(B_{\ell_p^n}) \sim \left(\frac{\log(n)}{n} \right)^{1 - \frac{1}{\min\{p,2\}}}.$$

The proof of the exact asymptotic behavior of $K(B_{\ell_p^n})$ given in [DF11] for $p < 2$, as mentioned before, use "sophisticated machinery" from the Banach space theory. Inspired by recent results from the general theory of Dirichlet series, in [BDS16] Bayart, Defant and Schlüters managed to give upper estimates for the unconditional basis constants of spaces of polynomials on ℓ_p spanned by finite sets of monomials, which avoid the use of this "machinery". This perspective gives a new and, in a sense, clear proof of Theorem 1.1 for the case $p < 2$.

The study of the Bohr radius together with the techniques developed for this purpose have been enriching many mathematical areas such us number theory [CDG$^+$15, DFOC$^+$11, BB04], complex analysis [BDF$^+$17, BDS16, DMP09], operator algebras [PPS02, Dix95], random polynomials [Boa00] (together with several works influenced by this article such as [DGM04, Bay12, GMSP97]), the study of functions on the boolean cube [DMP17a, DMP17b] (which are fundamental in theoretical computer science, graph theory, social choice, etc.) and even in quantum information [DMP17b, Mon12].

Our aim is to continue the study of the Bohr phenomenon for mixed Reinhardt domains. Let $\mathcal{R}$ and $\mathcal{S}$ be two Reinhardt domain in $\mathbb{C}^n$. The mixed Bohr radius $K(\mathcal{R}, \mathcal{S})$ is defined as the
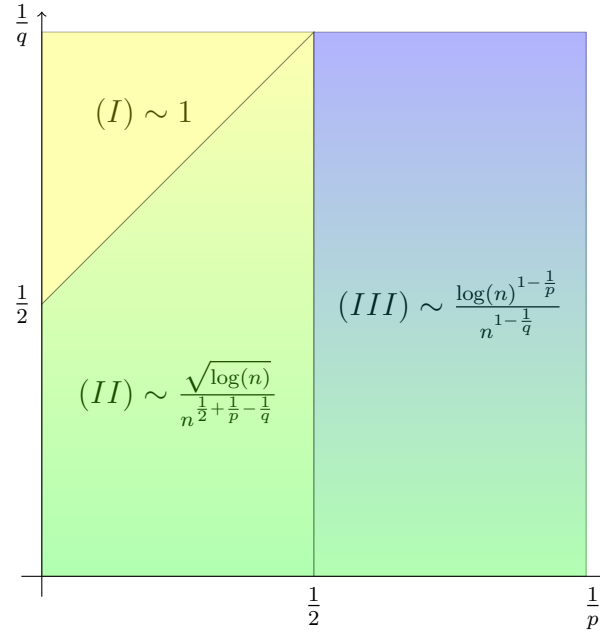
FIGURE 1. Graphical overview of the mixed Bohr radius described in Theorem 1.2.

biggest number $r \geq 0$ such that for every analytic function $f(z) = \sum_\alpha a_\alpha z^\alpha$ bounded on $\mathcal{R}$, it holds:

$$(5) \qquad \sup_{z \in r \cdot \mathcal{S}} \sum_\alpha |a_\alpha z^\alpha| \leq \sup_{z \in \mathcal{R}} |f(z)|.$$

We will focus in the case where $\mathcal{R}$ and $\mathcal{S}$ are the closed unit balls of ℓ_p and ℓ_q for $1 \leq p, q \leq \infty$. Note that $K(B_{\ell_p^n})$ in the previous notation is just $K(B_{\ell_p^n}, B_{\ell_p^n})$. Our contribution is the following theorem which provides the correct asymptotic estimates for the full range of p's and q's.

Theorem 1.2. *Let* $1 \leq p, q \leq \infty$*, with* $q \neq 1$*. The asymptotic growth of the* (p, q)*-Bohr radius is given by*

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \sim \begin{cases} 1 & \text{if (I): } 2 \leq p \leq \infty \ \wedge \ \frac{1}{2} + \frac{1}{p} \leq \frac{1}{q}, \\ \dfrac{\sqrt{\log(n)}}{n^{\frac{1}{2} + \frac{1}{p} - \frac{1}{q}}} & \text{if (II): } 2 \leq p \leq \infty \ \wedge \ \frac{1}{2} + \frac{1}{p} > \frac{1}{q}, \\ \dfrac{\log(n)^{1 - \frac{1}{p}}}{n^{1 - \frac{1}{q}}} & \text{if (III): } 1 \leq p, q \leq 2. \end{cases}$$

For $q = 1$ *and every* $1 \leq p \leq \infty$*,* $K(B_{\ell_p^n}, B_{\ell_q^n}) \sim 1$*.*

As for $K(B_{\ell_p^n})$, the upper bound are obtained using random polynomials with adequate coefficients and relatively small norm [Boa00, DGM04, Bay12]. To obtain the lower bounds the proof

is divided in several cases. For $p < 2$ we have combined an appropriate way to divide and distinguish certain subsets of monomials together with the upper estimates for the unconditional basis constants of spaces of polynomials on ℓ_p spanned by finite sets of monomials given in [BDS16]. The interplay between monomial convergence and mixed unconditionality for spaces of homogeneous polynomials presented in [DMP09, Theorem 5.1.] (which, of course, gives information on the Bohr radius) is crucial for the case $p > 2$. We have strongly used some recent inclusion for the set of monomial convergence $\mathrm{dom} H_\infty(B_{\ell_p})$ $p \geq 2$ given in [DMP09, BDF+17]. Therefore, it is worth noting that the techniques and results developed in the last years were fundamental for our proof.

The article is organized as follows. In Section 2 we present some basic background and results that we will use to prove Theorem 1.2. We also give in this section some of the notation and concepts that appeared in this introduction. In Sections 3 and 4 we show the upper and lower estimates for the theorem respectively.

2. Preliminaries

We write by $\mathbb{D}$ the closed unit disk in the complex plane $\mathbb{C}$. As usual we denote ℓ_p^n for the Banach space of all n-tuples $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ endowed with the norm $\|(z_1, \ldots, z_n)\|_p = \left(\sum_{i=1}^n |z_i|^p \right)^{1/p}$ if $1 \leq p < \infty$, and $\|(z_1, \ldots, z_n)\|_\infty = \max_{i=1,\ldots,n} |z_i|$ for $p = \infty$. The unit ball of ℓ_p^n is denoted by $B_{\ell_p^n}$. For $1 \leq p \leq \infty$ we write p' for its conjugate exponent (i.e., $\frac{1}{p} + \frac{1}{p'} = 1$).

For every $x, y \in \mathbb{C}^\mathbb{N}$ we denote $|x| = (|x_1|, \ldots, |x_n|, \ldots)$, and $|x| \leq |y|$ will mean that $|x_i| \leq |y_i|$ for every $i \in \mathbb{N}$. Recall that a Banach sequence space is a Banach space $(X, \|\cdot\|_X)$ with $\ell_1 \subset X \subset \ell_\infty$; and such that whenever $y \in X$, $x \in \mathbb{C}^\mathbb{N}$ and $|x| \leq |y|$ it follows $x \in X$ and $\|x\|_X \leq \|y\|_X$. A non-empty open set $\mathcal{R} \subset X$ is called a Reinhardt domain whenever given $x \in \mathbb{C}^\mathbb{N}$ and $y \in \mathcal{R}$ such that $|x| \leq |y|$ then it holds $x \in \mathcal{R}$.

Given a Banach sequence space X and fixed $n \in \mathbb{N}$ its n-th projection X_n is defined as the quotient space induced by the mapping

$$\pi_n : X \to \mathbb{C}^n$$

$$x \mapsto (x_1, \ldots, x_n).$$

An m-homogeneous polynomial in n variables is a function $P : \mathbb{C}^n \to \mathbb{C}$ of the form

$$P(z_1, \ldots, z_n) = \sum_{\alpha \in \Lambda(m,n)} a_\alpha z^\alpha,$$

where $\Lambda(m,n) := \{\alpha \in \mathbb{N}_0^n : |\alpha| := \alpha_1 + \cdots + \alpha_n = m\}$, $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $a_\alpha \in \mathbb{C}$. We will use the notation $a_\alpha =: a_\alpha(P)$.

Another way of writing a polynomial P is as follows:

$$P(z_1, \ldots, z_n) = \sum_{\mathbf{j} \in \mathcal{J}(m,n)} c_{\mathbf{j}} z_{\mathbf{j}},$$

where $\mathcal{J}(m,n) := \{\mathbf{j} = (j_1, \ldots, j_m) : 1 \leq j_1 \leq \ldots \leq j_m \leq n\}$, $z_{\mathbf{j}} := z_{j_1} \cdots z_{j_m}$ and $c_{\mathbf{j}} \in \mathbb{C}$. Note that $c_{\mathbf{j}} = a_\alpha$ with $\mathbf{j} = (1, \overset{\alpha_1}{\ldots}, 1, \ldots, n, \overset{\alpha_n}{\ldots}, n)$. For some fixed $\mathbf{j} \in \mathcal{J}(m,n)$ and some $\mathbf{i} = (i_1, \ldots, i_m) \in \mathbb{N}^m$ we say $\mathbf{i} \in [\mathbf{j}]$ if the exists some permutation $\sigma \in S_m$ such that $(i_{\sigma(1)}, \ldots, i_{\sigma(m)}) = \mathbf{j}$ and $|\mathbf{j}|$ will denote the number of elements in $[\mathbf{j}]$. Observe that $|\mathbf{j}| = \frac{m!}{\alpha!}$ if the index $\mathbf{j}$ is $(1, \overset{\alpha_1}{\ldots}, 1, \ldots, n, \overset{\alpha_n}{\ldots}, n)$.

The elements $(z^\alpha)_{\alpha \in \Lambda(m,n)}$ (equivalently, $(z_{\mathbf{j}})_{\mathbf{j} \in \mathcal{J}(m,n)}$) are commonly refereed as *the monomials*.

Following the notation of [BDS16], given a subset $\mathcal{J} \subset \mathcal{J}(m,n)$, we call

$$\mathcal{J}^* = \{\mathbf{j} \in \mathcal{J}(m-1, n) : \text{ there is } k \geq 1, (\mathbf{j}, k) \in \mathcal{J}\}.$$

For $1 \leq p \leq \infty$ we denote by $\mathcal{P}(^m \ell_p^n)$ the Banach space of all m-homogeneous polynomials in n complex variables equipped with the uniform (or sup) norm

$$\|P\|_{\mathcal{P}(^m \ell_p^n)} := \sup_{z \in B_{\ell_p^n}} |P(z)|.$$

Given two Banach sequence spaces X and Y, for $n, m \in \mathbb{N}$ let $\chi_M(\mathcal{P}(^m X_n), \mathcal{P}(^m Y_n))$ be the best constant $\lambda > 0$ such that

$$\sup_{z \in B_{Y_n}} \left| \sum_{\alpha \in \Lambda(m,n)} \theta_\alpha a_\alpha z^\alpha \right| \leq \lambda \sup_{z \in B_{X_n}} \left| \sum_{\alpha \in \Lambda(m,n)} a_\alpha z^\alpha \right|,$$

for every $(a_\alpha)_{\alpha \in \Lambda(m,n)} \subset \mathbb{C}$ and every choice of complex numbers $(\theta_\alpha)_{\alpha \in \Lambda(m,n)}$ of modulus one.

When $X = \ell_p$ and $Y = \ell_q$ we will denote $\chi_M(\mathcal{P}(^m X_n), \mathcal{P}(^m Y_n))$, the (p,q)-mixed unconditionally constant for the monomial basis of $\mathcal{P}(^m \mathbb{C}^n)$, as $\chi_M(\mathcal{P}(^m \ell_p^n), \mathcal{P}(^m \ell_q^n))$. It should be mentioned that, for any fixed $m \in \mathbb{N}$, the asymptotic growth of $\chi_M(\mathcal{P}(^m \ell_p^n), \mathcal{P}(^m \ell_q^n))$ as $n \to \infty$ was studied in [GMM16].

Every entire function $f : \mathbb{C}^n \to \mathbb{C}$ can be written as

$$f = \sum_{m \geq 0} \sum_{\alpha \in \Lambda(m,n)} a_\alpha(f) z^\alpha.$$

Recall that $K(B_{\ell_p^n}, B_{\ell_q^n})$ stands for the n-dimensional (p,q)-Bohr radius. That is, $K(B_{\ell_p^n}, B_{\ell_q^n})$ denotes the greatest constant $r > 0$ such that for every entire function $f = \sum_\alpha a_\alpha z^\alpha$ in n-complex

variables, we have the following (mixed) Bohr-type inequality

$$\sup_{z \in r \cdot B_{\ell_q^n}} \sum_{\alpha} |a_\alpha z^\alpha| \leq \sup_{z \in B_{\ell_p^n}} |f(z)|.$$

In the same way, the m-homogeneous mixed Bohr radius, $K_m(B_{\ell_p^n}, B_{\ell_q^n})$, is defined as the greatest $r > 0$ such that for every $P(z) = \sum_{\alpha \in \Lambda(m,n)} a_\alpha z^\alpha \in \mathcal{P}(^m\mathbb{C}^n)$ it follows

$$\sup_{z \in B_{\ell_q^n}} \sum_{\alpha \in \Lambda(m,n)} |a_\alpha z^\alpha| r^m = \sup_{z \in r \cdot B_{\ell_q^n}} \sum_{\alpha \in \Lambda(m,n)} |a_\alpha z^\alpha| \leq \|P\|_{\mathcal{P}(^m\ell_p^n)}.$$

It is plain that $K(B_{\ell_p^n}, B_{\ell_q^n}) \leq K_m(B_{\ell_p^n}, B_{\ell_q^n})$.

Remark 2.1.

$$K_m(B_{\ell_p^n}, B_{\ell_q^n}) = \frac{1}{\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}}.$$

Proof. Given $P \in \mathcal{P}(^m\mathbb{C}^n)$ and for any $(\theta_\alpha)_{\alpha \in \Lambda(m,n)}$ we have

$$\| \sum_{\alpha \in \Lambda(m,n)} \theta_\alpha a_\alpha(P) z^\alpha \|_{\mathcal{P}(^m\ell_q^n)} \leq \| \sum_{\alpha \in \Lambda(m,n)} |a_\alpha(P) z^\alpha| \|_{\mathcal{P}(^m\ell_q^n)}$$

$$= \| \sum_{\alpha \in \Lambda(m,n)} |a_\alpha(P) z^\alpha| (K_m(B_{\ell_p^n}, B_{\ell_q^n}))^m \|_{\mathcal{P}(^m\ell_q^n)} \frac{1}{(K_m(B_{\ell_p^n}, B_{\ell_q^n}))^m}$$

$$\leq \frac{1}{(K_m(B_{\ell_p^n}, B_{\ell_q^n}))^m} \|P\|_{\mathcal{P}(^m\ell_p^n)},$$

which leads to the inequality $\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m} \leq \frac{1}{K_m(B_{\ell_p^n}, B_{\ell_q^n})}$. On the other hand, for $P \in \mathcal{P}(^m\mathbb{C}^n)$ take $\theta_\alpha = \frac{\overline{a_\alpha(P)}}{|a_\alpha(P)|}$. Then we have

$$\| \sum_{\alpha \in \Lambda(m,n)} |a_\alpha(P) z^\alpha| \|_{\mathcal{P}(^m\ell_q^n)} = \| \sum_{\alpha \in \Lambda(m,n)} \theta_\alpha a_\alpha(P) z^\alpha \|_{\mathcal{P}(^m\ell_q^n)}$$

$$\leq \chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n)) \|P\|_{\mathcal{P}(^m\ell_p^n)},$$

or equivalently,

$$\sup_{z \in B_{\ell_q^n}} \sum_{\alpha \in \Lambda(m,n)} |a_\alpha z^\alpha| \left(\frac{1}{\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}} \right)^m \leq \|P\|_{\mathcal{P}(^m\mathbb{C}^n)},$$

which means $\frac{1}{\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}} \leq K_m(B_{\ell_p^n}, B_{\ell_q^n})$. □

It will be useful to remember a classic result due to F. Wiener (see [KB97]) which asserts that for every holomorphic function f written as the sum of m-homogeneous polynomials as $f = \sum_{m \geq 1} P_m + a_0$ and such that $\sup_{z \in B_{\ell_p^n}} |f(z)| \leq 1$ it holds

(6) $$\|P_m\|_{\mathcal{P}(^m\ell_p^n)} \leq 1 - |a_0|^2,$$

for every $m \in \mathbb{N}$.

In general this inequality is presented for the uniform norm on the polydisk $\| \cdot \|_{\mathcal{P}(^m\ell_\infty^n)}$ (i.e., $p = \infty$), but this version easily follows by a standard modification of the original argument (given $z \in B_{\ell_p^n}$ consider the auxiliary function $g : \mathbb{C}^n \to \mathbb{C}$ given by $g(w) := f(w \cdot z)$).

The next lemma is an adaption of the case $p = q$, see [DGM03, Theorem 2.2.] and constitutes the basic link between Bohr radius and unconditional basis constants of spaces of polynomials on the mixed context (p not necessarily equal to q).

Lemma 2.2. *For every $n \in \mathbb{N}$ and $1 \le p, q \le \infty$ it holds*

$$\frac{1}{3} \frac{1}{\sup_{m \ge 1} \chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}} \le K(B_{\ell_p^n}, B_{\ell_q^n}) \le \min \left\{ \frac{1}{3}, \frac{1}{\sup_{m \ge 1} \chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}} \right\}.$$

Proof. Form Remark 2.1 we have $K(B_{\ell_p^n}, B_{\ell_q^n}) \le \inf_{m \ge 1} \frac{1}{\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}} = \frac{1}{\sup_{m \ge 1} \chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}}$ and due to Bohr's inequality we know $K(\mathbb{D}) = \frac{1}{3}$ as it is clear that $K(B_{\ell_p^n}, B_{\ell_q^n}) \le K(\mathbb{D})$ for every $n \in \mathbb{N}$ the right hand side inequality holds. For the left hand side inequality let us take some holomorphic function f, without loss of generality let us assume $\sup_{z \in B_{\ell_p^n}} |f(z)| = 1$, and consider its decomposition as a sum of m-homogeneous polynomials $f = \sum_{m \ge 0} P_m$. For every $m \in \mathbb{N}_0$ it holds $P_m(z) = \sum_{\alpha \in \Lambda(m,n)} a_\alpha(f) z^\alpha$, thus taking $\rho = \sup_{m \ge 1} \chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}$ and using Remark 2.1 again it follows

$$\left\| \sum_{\alpha \in \Lambda(m,n)} |a_\alpha(f)| \left(\frac{z}{\rho} \right)^\alpha \right\|_{\mathcal{P}(^m\ell_q^n)} \le \left\| \sum_{\alpha \in \Lambda(m,n)} a_\alpha(f) z^\alpha \right\|_{\mathcal{P}(^m\ell_p^n)}.$$

Applying the above mentioned Wiener's result for some $w \in B_{\ell_q^n}$ we have that

$$\sum_{m \ge 0} \sum_{\alpha \in \Lambda(m,n)} |a_\alpha(f)| \left(\frac{w}{3\rho} \right)^\alpha \le |a_0(f)| + \sum_{m \ge 1} \frac{1}{3^m} \left\| \sum_{\alpha \in \Lambda(m,n)} a_\alpha(f) z^\alpha \right\|_{\mathcal{P}(^m\ell_p^n)}$$

$$\le |a_0(f)| + \sum_{m \ge 1} \frac{1}{3^m} (1 - |a_0(f)|^2)$$

$$\le |a_0(f)| + \frac{1 - |a_0(f)|^2}{2}$$

$$\le 1 = \sup_{z \in B_{\ell_p^n}} |f(z)|,$$

where last inequality holds as $|a_0(f)| \le \sup_{z \in B_{\ell_p^n}} |f(z)| = 1$. The last chain of inequalities and the maximality of mixed Bohr radius lead us to $\frac{1}{3\rho} \le K(B_{\ell_p^n}, B_{\ell_q^n})$ as we wanted to prove. $\square$
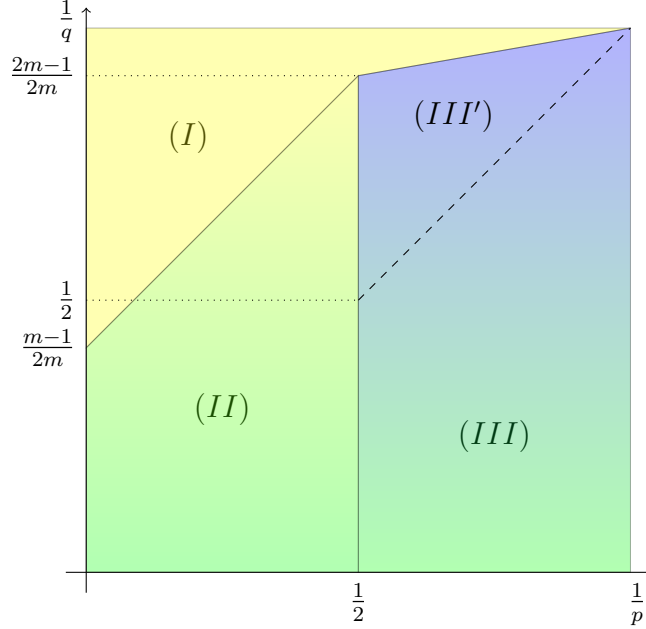
FIGURE 2. Graphical overview of the mixed unconditional constant described in Theorem 2.3.

The previous lemma shows that understanding $K(B_{\ell_p^n}, B_{\ell_q^n})$ translates into seeing how the constant $\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}$ behaves. Unfortunately, the results on the asymptotic growth of the (mixed) unconditional constant $\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))$ (for fixed m) as $n \to \infty$ from [GMM16] are not useful here. As can be seen in Lemma 2.2, one needs to comprehend how $\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}$ grows by moving both the number of variables, n, and the degree of homogeneity, m. But beyond this, they give a guideline of what to expect (at least what the different regions in Figure 1 should look like). Indeed, in [GMM16] we have proved the following

Theorem 2.3.
$$\begin{cases} \chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \sim 1 & \text{for } (I): \; [\frac{1}{p} + \frac{m-1}{2m} \leq \frac{1}{q} \wedge \frac{1}{p} \leq \frac{1}{2}] \text{ or } [\frac{m-1}{m} + \frac{1}{mp} < \frac{1}{q} \wedge \frac{1}{2} \leq \frac{1}{p}], \\ \chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \sim n^{m(\frac{1}{2}+\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} & \text{for } (II) \; [\frac{1}{p} + \frac{m-1}{2m} \geq \frac{1}{q} \wedge \frac{1}{p} \leq \frac{1}{2}], \\ \chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \sim n^{(m-1)(1-\frac{1}{q})+\frac{1}{p}-\frac{1}{q}} & \text{for } (III) \;: \; [\frac{1}{p} \geq \frac{1}{q} \wedge \frac{1}{2} \leq \frac{1}{p}], \\ \chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \sim_\varepsilon n^{(m-1)(1-\frac{1}{q})+\frac{1}{p}-\frac{1}{q}} & \text{for } (III') \;: \; [1 - \frac{1}{m} + \frac{1}{mp} \geq \frac{1}{q} \geq \frac{1}{p} \wedge \frac{1}{2} < \frac{1}{p} < 1]. \end{cases}$$
where $\chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \sim_\varepsilon n^{(m-1)(1-\frac{1}{q})+\frac{1}{p}-\frac{1}{q}}$ means that
$$n^{(m-1)(1-\frac{1}{q})+\frac{1}{p}-\frac{1}{q}} \ll \chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \ll n^{(m-1)(1-\frac{1}{q})+\frac{1}{p}-\frac{1}{q}+\varepsilon},$$
for every $\varepsilon > 0$.

The heuristic to interpret the different regions in Theorem 1.2 is the following: If one thinks that the correct order of the region (III') coincides with that of (III) in Theorem 2.3 (in fact,

this is what we believe) and assumes that the homogeneity degree is very very large ($m \to \infty$) then the graph in Figure 2 transforms into the one presented in Figure 1. All this, together with the upper bounds that one gets after using classical random polynomials (see Section 3, somehow the easy part) helped us to define where to aim to prove lower bounds. We highlight that, the logarithmic factors that appear in Theorem 1.2, are missing in Theorem 2.3. This is, somehow, not coincidental and their presence is due to the interplay between the number of variables and the degree of homogeneity in Lemma 2.2.

We continue with some definitions that will be useful later. Given X a Banach sequence space we denote $H_\infty(B_X)$ to the space of holomorphic functions over B_X endowed with the norm given by $\|f\|_{H_\infty(B_X)} := \sup_{z \in B_X} |f(z)|$. The domain of monomial convergence of $H_\infty(B_X)$ is defined as

$$\mathrm{dom}H_\infty(B_X) := \Big\{ z \in \ell_\infty : \sum_{m \geq 0} \sum_{\alpha \in \Lambda(m,n)} |a_\alpha(f)z^\alpha| < \infty \text{ for every } f \in H_\infty(B_X) \Big\}.$$

The next theorem appears in [DMP09, Theorem 5.1] and relates monomial convergence with the study of the mixed unconditional constant of the monomial basis.

Theorem 2.4. *For a couple of Banach sequence spaces X, Y the following are equivalent:*

(1) $rB_Y \subset domH_\infty(B_X)$ *for some* $r > 0$.
(2) *There exists a constant* $C > 0$ *independent of m such that*

$$\sup_{n \geq 1} \chi_M(\mathcal{P}(^m X_n), \mathcal{P}(^m Y_n)) \leq C^m.$$

If $(a_n)_n$ and $(b_n)_n$ are two sequences of real numbers we will write $a_n \ll b_n$ if there exists a constant $C > 0$ (independent of n) such that $a_n \leq Cb_n$ for every n. We will write $a_n \sim b_n$ if $a_n \ll b_n$ and $b_n \ll a_n$. We will use repeatedly the Stirling formula which asserts

(7) $$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.$$

We use the letters C, C_1, C_2, etc. to denote certain constants which do not depend on n (the number of variables) nor m (the homogeneity degree). These may depend on the parameters p and/or q and from one inequality to the other may vary (and will be denoted in the same way).

3. Upper bounds

Upper bounds constitute the easy part: we will use the classical probabilistic approach. Bayart in [Bay12, Corollary 3.2] (see also [Boa00, DGM03, DGM04]) exhibited polynomials with small

sup-norm on the unit ball of ℓ_p^n. He showed that for each $1 \leq p \leq \infty$ there exists an m-homogeneous polynomial in n complex variables, $P(z) := \sum_{\alpha \in \Lambda(m,n)} \varepsilon_\alpha \frac{m!}{\alpha!} z^\alpha$, with $\varepsilon_\alpha = \pm 1$ for every α, such that $\|P\|_{\mathcal{P}(^m \ell_p^n)} \leq D_p(m,n)$ where

$$(8) \qquad D_p(m,n) := C_p \times \begin{cases} (\log(m) m!)^{1 - \frac{1}{p}} \, n^{1 - \frac{1}{p}} & \text{if } 1 \leq p \leq 2, \\ (\log(m) m!)^{\frac{1}{2}} n^{m(\frac{1}{2} - \frac{1}{p}) + \frac{1}{2}} & \text{if } 2 \leq p \leq \infty, \end{cases}$$

and C_p depends exclusively on p.

We will also need the following remark which is an easy calculus exercise.

Remark 3.1. For every positive numbers $a, b > 0$ and $n \in \mathbb{N}$, the function $f : \mathbb{R}_{>0} \to \mathbb{R}$ given by $f(x) = x^a n^{\frac{b}{x}}$ attains its minimum at $x = \log(n) \frac{b}{a}$.

Proof of the upper bounds of Theorem 1.2. Upper bounds for the case $\frac{1}{p} + \frac{1}{2} \leq \frac{1}{q}$ and $q = 1$ in Theorem 1.2 are trivial.

Suppose now $\frac{1}{p} + \frac{1}{2} > \frac{1}{q}$ and let $(\varepsilon_\alpha)_{\alpha \in \Lambda(m,n)} \subset \{-1, 1\}$ signs such that

$$\left\| \sum_{\alpha \in \Lambda(m,n)} \varepsilon_\alpha \frac{m!}{\alpha!} z^\alpha \right\|_{B_{\ell_p^n}} \leq D_p(m,n),$$

as (8). Taking $z_0 = (\frac{1}{n^{1/q}}, \dots, \frac{1}{n^{1/q}}) \in B_{\ell_q}$ we can conclude that

$$n^{m(1 - \frac{1}{q})} = \sum_{\alpha \in \Lambda(m,n)} \frac{m!}{\alpha!} \left(\frac{1}{n^{\frac{1}{q}}} \right)^m$$

$$\leq \left\| \sum_{\alpha \in \Lambda(m,n)} |\varepsilon_\alpha| \frac{m!}{\alpha!} z^\alpha \right\|_{B_{\ell_q^n}}$$

$$\leq \chi_M(\mathcal{P}(^m \ell_p^n), \mathcal{P}(^m \ell_q^n)) \left\| \sum_{\alpha \in \Lambda(m,n)} \varepsilon_\alpha \frac{m!}{\alpha!} z^\alpha \right\|_{B_{\ell_p^n}}$$

$$\leq \chi_M(\mathcal{P}(^m \ell_p^n), \mathcal{P}(^m \ell_q^n)) \cdot D_p(m,n).$$

We have by Stirling formula (7) and for $p < 2$,

$$\frac{1}{\chi_M(\mathcal{P}(^m \ell_p^n), \mathcal{P}(^m \ell_q^n))^{1/m}} \leq \left(C n^{\frac{1}{p'}} (\log(m) m!)^{\frac{1}{p'}} \right)^{1/m} \frac{1}{n^{\frac{1}{q'}}}$$

$$\leq C \frac{1}{n^{\frac{1}{q'}}} m^{\frac{1}{p'}} n^{\frac{1}{p'm}}.$$

Thanks to Lemma 2.1, Remark 3.1 and the previous inequality

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \leq C \frac{1}{n^{\frac{1}{q'}}} \inf_{m \geq 1} m^{\frac{1}{p'}} n^{\frac{1}{p'm}} \leq C \frac{\log(n)^{\frac{1}{p'}}}{n^{\frac{1}{q'}}}.$$

On the other hand, for $p \geq 2$ and $\frac{1}{p} + \frac{1}{2} > \frac{1}{q}$ it follows

$$\frac{1}{\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n)))^{1/m}} \leq \left(Cn^{\frac{1}{2}} \left(\log(m)m!\right)^{\frac{1}{2}}\right)^{1/m} \frac{1}{n^{\frac{1}{2} + \frac{1}{p} - \frac{1}{q}}}$$

$$\leq C \frac{1}{n^{\frac{1}{2} + \frac{1}{p} - \frac{1}{q}}} m^{\frac{1}{2}} n^{\frac{1}{2m}}.$$

Thus minimizing $m^{\frac{1}{2}} n^{\frac{1}{2m}}$ as in the previous case we get,

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \leq C \frac{\sqrt{\log(n)}}{n^{\frac{1}{2} + \frac{1}{p} - \frac{1}{q}}},$$

as we wanted to prove. □

4. Lower bounds

For the proof of the lower bounds we need to consider four different cases. We begin with the case $q = 1$ and the case $p \leq q$, which are the easy ones. Then we study the case $1 < q \leq p \leq 2$ where we use tools from unconditionality and finally the case $p \geq 2$ where the key tool is monomial convergence.

4.1. **The case $q = 1$.** By [Aiz00], $K(B_{\ell_1^n}) \sim 1$. Thus, for any $f(z) = \sum_\alpha a_\alpha z^\alpha$, it follows that

$$\sup_{z \in K(B_{\ell_1^n}) \cdot B_{\ell_1^n}} \sum_\alpha |a_\alpha z^\alpha| \leq \sup_{z \in B_{\ell_1^n}} |f(z)| \leq \sup_{z \in B_{\ell_p^n}} |f(z)|,$$

which implies that $K(B_{\ell_p^n}, B_{\ell_1^n}) \geq K(B_{\ell_1^n}) \sim 1$.

4.2. **The case $p \leq q$.** For this case we will strongly use Theorem 1.1. The case $p \leq q$ is an easy corollary of this result. For any $P(z) = \sum_{\alpha \in \Lambda(m,n)} a_\alpha z^\alpha$, it follows that

$$\left\| \sum_{\alpha \in \Lambda(m,n)} |a_\alpha| z^\alpha \right\|_{\mathcal{P}(^m\ell_q^n)} \leq n^{m(\frac{1}{p} - \frac{1}{q})} \left\| \sum_{\alpha \in \Lambda(m,n)} |a_\alpha| z^\alpha \right\|_{\mathcal{P}(^m\ell_p^n)}$$

$$\leq n^{m(\frac{1}{p} - \frac{1}{q})} K(B_{\ell_p^n})^{-m} \left\| \sum_{\alpha \in \Lambda(m,n)} a_\alpha z^\alpha \right\|_{\mathcal{P}(^m\ell_p^n)},$$

which implies that $K_m(B_{\ell_p^n}, B_{\ell_q^n}) \geq K(B_{\ell_p^n}) n^{\frac{1}{q} - \frac{1}{p}}$ for every $m \in \mathbb{N}$. Using Lemma 2.2 and Theorem 1.1 we have, for $p \leq 2$,

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \geq \frac{1}{3} n^{\frac{1}{q} - \frac{1}{p}} K(B_{\ell_p^n}) \sim n^{\frac{1}{q} - \frac{1}{p}} \left(\frac{\log(n)}{n}\right)^{1 - \frac{1}{p}} = \frac{\log(n)^{1 - 1/p}}{n^{1 - 1/q}},$$

and, for $p \geq 2$,

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \geq \frac{1}{3} n^{\frac{1}{q} - \frac{1}{p}} K(B_{\ell_p^n}) \sim n^{\frac{1}{q} - \frac{1}{p}} \left(\frac{\log(n)}{n}\right)^{1 - \frac{1}{2}} = \frac{\sqrt{\log(n)}}{n^{\frac{1}{2} + \frac{1}{p} - \frac{1}{q}}}.$$

This concludes the proof.

4.3. **The case** $1 < q < p \leq 2$**.** The following lemma is a key tool for the results of this section. It provides $\ell_{p'}$-summability of certain coefficients of a given polynomial in terms of its uniform norm on ℓ_p^n.

Lemma 4.1. [BDS16, Lemma 3.5.] *Let* $1 \leq p \leq \infty$ *and* P *be an* m-*homogeneous polynomial in* n *variables. Then for any* $\mathbf{j} \in \mathcal{J}(m-1, n)$

$$\left(\sum_{k=j_{m-1}}^{n} |c_{(\mathbf{j},k)}(P)|^{p'} \right)^{1/p'} \leq m e^{1+\frac{m-1}{p}} |\mathbf{j}|^{1/p} \|P\|_{\mathcal{P}(^m \ell_p^n)}.$$

The next lemma is an adaptation of the case $r \leq 2$ in [BDS16, Theorem 3.2] to the mixed context for our purposes.

Lemma 4.2. *Let* $1 \leq q < p \leq 2$*. Then we have*

$$\chi_M(\mathcal{P}(^m \ell_p^n), \mathcal{P}(^m \ell_q^n)) \leq m e^{1+\frac{m-1}{p}} \left(\sum_{\mathbf{j} \in \mathcal{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'}.$$

Proof. Fix $P \in \mathcal{P}(^m \ell_p^n)$ and $u \in \ell_q^n$. Then, by Lemma 4.1, for any $\mathbf{j} \in \mathcal{J}(m, n)^*$,

$$\left(\sum_{k: \, (\mathbf{j},k) \in \mathcal{J}(m,n)} |c_{(\mathbf{j},k)}(P)|^{p'} \right)^{1/p'} = \left(\sum_{k=j_{m-1}}^{n} |c_{(\mathbf{j},k)}(P)|^{p'} \right)^{1/p'} \leq m e^{1+\frac{m-1}{p}} |\mathbf{j}|^{1/p} \|P\|_{\mathcal{P}(^m \ell_p^n)}.$$

Now applying the above inequality, Hölder's inequality (two times) and the multinomial formula we have

$$
\begin{aligned}
\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P)||u_{\mathbf{j}}| &= \sum_{\mathbf{j}\in\mathcal{J}(m,n)^*} \left(\sum_{k:\ (\mathbf{j},k)\in\mathcal{J}(m,n)} |c_{(\mathbf{j},k)}||u_{\mathbf{j}}||u_k| \right) \\
&\leq \sum_{\mathbf{j}\in\mathcal{J}(m,n)^*} |u_{\mathbf{j}}| \left(\sum_{k:\ (\mathbf{j},k)\in\mathcal{J}(m,n)} |c_{(\mathbf{j},k)}|^{q'} \right)^{1/q'} \left(\sum_k |u_k|^q \right)^{1/q} \\
&\leq \sum_{\mathbf{j}\in\mathcal{J}(m,n)^*} |u_{\mathbf{j}}| \left(\sum_{k:\ (\mathbf{j},k)\in\mathcal{J}(m,n)} |c_{(\mathbf{j},k)}|^{p'} \right)^{1/p'} \|u\|_q \\
&\leq m e^{1+\frac{m-1}{p}} \sum_{\mathbf{j}\in\mathcal{J}(m,n)^*} |\mathbf{j}|^{1/p} |u_{\mathbf{j}}| \|u\|_q \|P\|_{\mathcal{P}(^m\ell_p^n)} \\
&\leq m e^{1+\frac{m-1}{p}} \left(\sum_{\mathbf{j}\in\mathcal{J}(m,n)^*} |\mathbf{j}||u_{\mathbf{j}}|^q \right)^{1/q} \left(\sum_{\mathbf{j}\in\mathcal{J}(m,n)^*} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \|u\|_q \|P\|_{\mathcal{P}(^m\ell_p^n)} \\
&\leq m e^{1+\frac{m-1}{p}} \left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,n)} |\mathbf{j}||u_{\mathbf{j}}|^q \right)^{1/q} \left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \|u\|_q \|P\|_{\mathcal{P}(^m\ell_p^n)} \\
&= m e^{1+\frac{m-1}{p}} \left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \|u\|_q^m \|P\|_{\mathcal{P}(^m\ell_p^n)},
\end{aligned}
$$

which gives the desired inequality. $\qquad\square$

The key to prove the lower bound is to obtain good bounds for the sum on the right hand side of the previous lemma.

Lemma 4.3. *Let $1 < q \leq p \leq 2$, for large enough n it follows*

$$
\left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \leq C^m \frac{n^{m/q'}}{\log(n)^{m/p'}},
$$

for every $m \in \mathbb{N}$.

With this at hand it is easy to prove the remaining lower bounds of the case $p \leq 2$ of the main theorem.

Proof of the lower bound of the case $1 < q < p \leq 2$ on Theorem 1.2. Thanks to Lemma 2.2 it is enough to prove that

(9) $$\frac{\log(n)^{1-1/p}}{n^{1-1/q}} \ll \frac{1}{\sup_{m \geq 1} \chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}} = \inf_{m \geq 1} \frac{1}{\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}},$$

which follows by Lemma 4.2 and Lemma 4.3. $\qquad\qquad\square$

The proof of Lemma 4.3 will require some hard work. We define for any $1 \leq k \leq m$ the k-bounded index set as

$$\Lambda_k(m, n) = \{\alpha \in \Lambda(m, n) : \alpha_i \leq k \text{ for all } 1 \leq i \leq n\}.$$

Let F be the bijective mapping connecting $\Lambda(m, n)$ and $\mathcal{J}(m, n)$ defined as

$$F : \mathcal{J}(m, n) \to \Lambda(m, n)$$

$$\mathbf{j} \mapsto \alpha$$

where $\alpha_i = F(\mathbf{j})_i = \#\{k : j_k = i\}$ for every $1 \leq i \leq n$. We denote

$$\mathcal{J}_k(m, n) = F^{-1}(\Lambda_k(m, n)),$$

for the corresponding k-bounded index set seen in $\mathcal{J}(m, n)$. Observe that for any $1 \leq k \leq m$ and $\mathbf{j} \in \mathcal{J}_k(m, n)$ the following inequalities hold:

(10) $$|\mathbf{j}| \geq \frac{m!}{k!^{\lceil \frac{m}{k} \rceil}} \geq \frac{m!}{k!^{\frac{m}{k}+1}} \geq C^m \frac{m^m}{k^{m+k}},$$

(11)
$$\left(\sum_{\mathbf{j} \in \mathcal{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \leq m^{1/q'} \max_{k=1,\dots,m-1} \left\{ \left(\sum_{\mathbf{j} \in \mathcal{J}_k(m-1,n) \cap \mathcal{J}_{k-1}^c(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \right\}.$$

Finally,

(12) $$|\mathcal{J}_{k-1}^c(m-1, n)| \leq n|\mathcal{J}(m-k-1, n)|$$

$$\leq n \binom{n+m-k-2}{m-k-1} \leq n\frac{(n+m-k-2)^{m-k-1}}{(m-k-1)!},$$

since $\mathbf{j} \in \mathcal{J}_{k-1}^c(m-1, n)$ requires that at least one of the variables is at the power of k (note that this bound is excessive, given that there are many repetitions at the time of counting, but it will be adequate for our purposes). For the particular case $m \leq n$ we can extract from inequality (12) the fact that

(13) $$|\mathcal{J}_{k-1}^c(m-1, n)| \leq 2^m \frac{n^{m-k}}{(m-k-1)!}.$$

We are now ready to prove Lemma 4.3.

Proof of Lemma 4.3. Suppose that n is large enough in order to have

(14) $$\log(n)^c \leq n,$$

where $c > 0$ is a constant that will be specified later.

We will split the proof in two cases.

- (i) $m \leq \log(n)^{\frac{q'}{p'}}$.

Note that if we take in (14) $c \geq \frac{q'}{p'}$, then $m \leq n$. Thus, by (10) and (13), we have, for each $1 \leq k \leq m - 1$,

$$\left(\sum_{\mathbf{j} \in \mathcal{J}_k(m-1,n) \cap \mathcal{J}_{k-1}^c(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \leq C^m |\mathcal{J}_{k-1}^c(m-1,n)|^{1/q'} \frac{k^{(m+k)(\frac{1}{q}-\frac{1}{p})}}{m^{m(\frac{1}{q}-\frac{1}{p})}}$$

$$\leq C^m \left(\frac{n^{m-k}}{(m-k-1)!} \right)^{\frac{1}{q'}} \frac{k^{(m+k)(\frac{1}{q}-\frac{1}{p})}}{m^{m(\frac{1}{q}-\frac{1}{p})}}.$$

Thus, by (11), we will prove the lemma if we are able to show that this last expression is $\leq C^m \frac{n^{m/q'}}{\log(n)^{m/p'}}$, for some constant $C > 0$. Therefore, it suffices to prove that, if $\beta := (\frac{1}{q} - \frac{1}{p})q'$,

(15) $$\frac{k^{\beta(m+k)}}{(m-k-1)! m^{\beta m}} \leq C^m \frac{n^k}{\log(n)^{m(\beta+1)}}.$$

Let us first suppose that $k \geq \min\{m/2, \frac{m}{3\beta}\} = dm$ for some $0 < d < 1$. Note that the left hand side is less than or equal to $m^{\beta m}$, which is $\leq \frac{n^{dm}}{\log(n)^{m(\beta+1)}}$ if we choose $cd > \beta(\frac{q'}{p'} + 1) + 1$ in (14) because $m \leq \log(n)^{\frac{q'}{p'}}$. Thus we have (15) for $k \geq dm$.

For $k \leq dm$, (15) is equivalent to

(16) $$\frac{k^{\beta(m+k)}}{m^{m-k} m^{\beta m}} \leq C^m \frac{n^k}{\log(n)^{m(\beta+1)}},$$

for some constant C. If $k = 1$ (16) is trivially satisfied. Let $1 < k \leq dm$. Using elementary calculus, it can be seen that $k^{\beta k} m^k / k^m \leq k^{m/3} m^k / k^m = m^k / k^{2m/3} \ll 1$ for $k \leq m/2$. Thus, it is enough to show that

$$\frac{k^{(\beta+1)m}}{m^{(\beta+1)m}} \leq C^m \frac{n^k}{\log(n)^{m(\beta+1)}}.$$

Taking logarithms, we see that it suffices to prove that, for some constant $C > 0$ we have,

(17) $$f(\frac{k \log(n)}{m}) - \frac{k \log(n)}{m} \leq C,$$

where $f(t) = (\beta + 1)\log(t)$. Note that there is some $t_0 = t_0(\beta) \geq 1$ such that $f(t) \leq t^{1/2}$ for every $t \geq t_0$. If $\frac{k\log(n)}{m} \geq t_0$ then $f(\frac{k\log(n)}{m}) - \frac{k\log(n)}{m} \leq \sqrt{\frac{k\log(n)}{m}} - \frac{k\log(n)}{m} \leq 0$, and (17) is satisfied. On the other hand, if $\frac{k\log(n)}{m} \leq t_0$ then (17) is fulfilled taking $C = f(t_0)$.

- (ii) For $m \geq \log(n)^{\frac{q'}{p'}}$ we just bound $|\mathbf{j}|^{(1/p-1/q)q'}$ by 1, thus we have by Stirling formula,

$$
\left(\sum_{\mathbf{j} \in \mathcal{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \leq |\mathcal{J}(m-1,n)|^{1/q'}
$$

$$
= \left(\frac{(n+m-2)!}{(m-1)!(n-1)!} \right)^{1/q'}
$$

$$
\leq C^m \left(\left(1 + \frac{n}{m-1} \right)^{m-1} \right)^{1/q'}
$$

$$
\leq C^m \left(1 + \frac{n}{\log(n)^{\frac{q'}{p'}}} \right)^{\frac{m-1}{q'}}
$$

$$
\leq C^m \frac{n^{\frac{m}{q'}}}{\log(n)^{\frac{m}{p'}}}.
$$

$\square$

4.4. **The case $p \geq 2$.** For the remaining cases it will be crucial the monomial convergence point of view from [BDF$^+$17, DMP09].

Proof of the case $\frac{1}{2} + \frac{1}{p} \leq \frac{1}{q}$ on Theorem 1.2. As a consequence of [DMP09, Example 4.9 (2)] we have that whenever $\frac{1}{q} \geq \frac{1}{r} = \frac{1}{p} + \frac{1}{2}$ and $p \geq 2$ it follows

$$
\ell_r \cap B_{\ell_p} \subset \mathrm{dom} H_\infty(B_{\ell_p}).
$$

Since $q \leq r \leq p$, then $B_{\ell_q} \subset B_{\ell_r} \subset B_{\ell_p}$ and then $B_{\ell_q} \subset \ell_r \cap B_{\ell_p} \subset \mathrm{dom} H_\infty(B_{\ell_p})$. Finally by Theorem 2.4 we have that there is some constant $C = C(p,q) > 0$ such that for every $n \in \mathbb{N}$ and p, q fulfilling the previous conditions it holds

$$
\frac{1}{\sup\limits_{m \geq 1} \left(\chi_M(\mathcal{P}(^m \ell_p^n), \mathcal{P}(^m \ell_q^n)) \right)^{1/m}} \geq C.
$$

As $K(B_{\ell_p^n}, B_{\ell_q^n}) \leq 1$ for every $1 \leq p, q \leq \infty$, the previous inequality and Lemma 2.2 lead us to the assertion that for $\frac{1}{q} \geq \frac{1}{p} + \frac{1}{2}$

(18) $$ K(B_{\ell_p^n}, B_{\ell_q^n}) \sim 1. $$

$\square$

For every $z \in \ell_\infty$ we can define $z^* \in \ell_\infty$ the decreasing rearrangement such that $|z_n^*| \geq |z_{n+1}^*|$ for every $n \in \mathbb{N}$. In [BDF$^+$17, Theorem 2.2] the authors proved that

$$B_\infty := \left\{ z \in \ell_\infty : \limsup_{n \to \infty} \frac{1}{\sqrt{\log(n)}} \left(\sum_{j=1}^n |z_j^*|^2 \right)^{1/2} < 1 \right\} \subset \mathrm{dom} H_\infty(B_{\ell_\infty}).$$

Consider now the space

$$X_\infty = \left\{ z \in \ell_\infty : \sup_{n \geq 2} \frac{1}{\sqrt{\log(n)}} \left(\sum_{j=1}^n |z_j^*|^2 \right)^{1/2} < \infty \right\},$$

endowed with the norm $\|z\|_{X_\infty} = \sup_{n \geq 2} \frac{1}{\sqrt{\log(n)}} \left(\sum_{j=1}^n |z_j^*|^2 \right)^{1/2}$. It is not difficult to see that this is Banach sequence space. Observe also that

(19) $$B_{X_\infty} \subset B_\infty \subset \mathrm{dom} H_\infty(B_{\ell_\infty}).$$

By Theorem 2.4 and expression (19) we have for some $C = C(p) > 0$

(20) $$\sup_{n \geq 2} \chi_M(\mathcal{P}(^m\ell_\infty^n), \mathcal{P}(^m(X_\infty)_n)) \leq C^m.$$

Note that the norm in $(X_\infty)_n$ is given by

$$\|(z_1, \ldots, z_n)\|_{(X_\infty)_n} = \sup_{2 \leq k \leq n} \frac{1}{\sqrt{\log(k)}} \left(\sum_{j=1}^k |z_j^*|^2 \right)^{1/2}.$$

To complete the study of the mixed Bohr radius for $p \geq 2$ it remains to understand the case $\frac{1}{q} < \frac{1}{2} + \frac{1}{p}$.

Remark 4.4. For every Reinhardt domain $\mathcal{R} \subset \mathbb{C}^n$, if $P \in \mathcal{P}(^m\mathbb{C}^n)$ and $w \in \mathcal{R}$ if we define $P_w \in \mathcal{P}(^m\mathbb{C}^n)$ as $P_w(z) = P(w \cdot z)$, it follows

$$\|P_w\|_{\mathcal{P}(^m\ell_\infty^n)} \leq \sup_{z \in \mathcal{R}} |P(z)|,$$

and $a_\alpha(P_w) = a_\alpha(P)w^\alpha$.

Proof of the case $\frac{1}{q} < \frac{1}{2} + \frac{1}{p}$ and $p \geq 2$ on Theorem 1.2. Fix $m \in \mathbb{N}$ and take $P \in \mathcal{P}(^m\mathbb{C}^n)$, $P(z) = \sum_{\alpha \in \Lambda(m,n)} a_\alpha z^\alpha$. By Lemma 2.2, it suffices to show that there exists some $C > 0$ such that for every $z \in B_{\ell_q^m}$ it holds

$$\sum_{\alpha \in \Lambda(m,n)} |a_\alpha z^\alpha| \leq C^m \left(\frac{n^{\frac{1}{2} + \frac{1}{p} - \frac{1}{q}}}{\sqrt{\log(n)}} \right)^m \|P\|_{\mathcal{P}(^m\ell_p^n)}.$$

Consider now $y = (z_1^{\frac{p}{p+2}}, \ldots, z_n^{\frac{p}{p+2}})$ and $w = (z_1^{\frac{2}{p+2}}, \ldots, z_n^{\frac{2}{p+2}})$. It is easy to see that $z = y \cdot w = (y_1 w_1, \ldots, y_n w_n)$, and thus, by (20) and Remark 4.4, we have

$$
\sum_{\alpha \in \Lambda(m,n)} |a_\alpha z^\alpha| = \sum_{\alpha \in \Lambda(m,n)} |a_\alpha w^\alpha y^\alpha|
$$
$$
\leq C^m \|y\|_{(X_\infty)_n}^m \|P_w\|_{\mathcal{P}(^m \ell_\infty^n)}
$$
$$
\leq C^m \|y\|_{(X_\infty)_n}^m \|w\|_{\ell_p^n}^m \|P\|_{\mathcal{P}(^m \ell_p^n)}.
$$

It remains to check that

$$
\|y\|_{(X_\infty)_n} \|w\|_{\ell_p^n} \leq C \frac{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}}{\sqrt{\log(n)}}.
$$

To start, let $1 \leq k \leq n$ then

$$
\|(y_1^*, \ldots, y_k^*)\|_{\ell_2^k} = \|(z_1^*, \ldots, z_k^*)\|_{\ell_{\frac{2p}{p+2}}^k}^{\frac{p}{p+2}}
$$
$$
\leq \left(\|(z_1^*, \ldots, z_k^*)\|_{\ell_q^k} k^{\frac{1}{p}+\frac{1}{2}-\frac{1}{q}} \right)^{\frac{p}{p+2}}
$$
$$
\leq \|z\|_{\ell_q^n}^{\frac{p}{p+2}} \left(k^{\frac{1}{p}+\frac{1}{2}-\frac{1}{q}} \right)^{\frac{p}{p+2}},
$$

so we have

$$
\|y\|_{(X_\infty)_n} = \sup_{2 \leq k \leq n} \frac{1}{\sqrt{\log(k)}} \|(y_1^*, \ldots, y_k^*)\|_{\ell_2^k}
$$
$$
\leq \sup_{2 \leq k \leq n} \frac{1}{\sqrt{\log(k)}} \|z\|_{\ell_q^n}^{\frac{p}{p+2}} \left(k^{\frac{1}{p}+\frac{1}{2}-\frac{1}{q}} \right)^{\frac{p}{p+2}}
$$
$$
\leq C \|z\|_{\ell_q^n}^{\frac{p}{p+2}} \frac{n^{\frac{1}{2}-\frac{1}{q}\frac{p}{p+2}}}{\sqrt{\log(n)}}.
$$

On the other hand,

$$
\|w\|_{\ell_p^n} = \|z\|_{\ell_{\frac{2p}{p+2}}^n}^{\frac{2}{2+p}}
$$
$$
\leq \|z\|_{\ell_q^n}^{\frac{2}{2+p}} n^{\left(\frac{1}{2}+\frac{1}{p}-\frac{1}{q}\right)\frac{2}{2+p}}
$$
$$
= \|z\|_{\ell_q^n}^{\frac{2}{2+p}} n^{\frac{1}{p}-\frac{1}{q}\frac{2}{2+p}}.
$$

Finally,

$$
\|y\|_{(X_\infty)_n} \|w\|_{\ell_p^n} \leq C \|z\|_{\ell_q^n}^{\frac{p}{p+2}} \frac{1}{\sqrt{\log(n)}} n^{\frac{1}{2}-\frac{1}{q}\frac{p}{p+2}} \|z\|_{\ell_q^n}^{\frac{2}{2+p}} n^{\frac{1}{p}-\frac{1}{q}\frac{2}{2+p}}
$$
$$
= C \|z\|_{\ell_q^n} \frac{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}}{\sqrt{\log(n)}},
$$

as we needed.   □

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