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Monomial decomposition and summability for holomorphic functions in high dimensions

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Descomposición monomial y sumabilidad para funciones holomorfas en altas dimensiones

Resumen

El objetivo de esta tesis es contribuir a la teoría de funciones holomorfas y polinomios homogéneos en varias e infinitas variables. Estudiamos diversos objetos que, de una u otra manera, involucran la sumabilidad de los coeficientes de polinomios homogéneos dependiendo de su norma uniforme.

Comparamos las normas uniforme y de coeficientes en espacios de polinomios homogéneos en varias variables complejas. En particular estudiamos el comportamiento asintótico de las constantes de equivalencia entre estas dos normas cuando la cantidad de variables tiende a infinito.

Damos una descripción completa del comportamiento asintótico de las constantes de incondicionalidad mixtas en espacios de polinomios homogéneos en finitas variables. Para lograrlo resulta indispensable el estudio que hacemos de los conjuntos de convergencia monomial para estos espacios de polinomios. En este sentido conseguimos un progreso sustancial en la caracterización de dichos conjuntos para el caso de polinomios homogéneos en ℓ_r cuando $1 < r \leqslant 2$, probando una conjetura abierta en el área.

Introducimos novedosas descomposiciones en el conjunto de monomios, que son de gran utilidad para atacar problemas de incondicionalidad y sumabilidad permitiendo un manejo adecuado de la dependencia entre el grado de homogeneidad y la cantidad de variables en ciertas desigualdades.

Definimos el radio de Bohr mixto extendiendo la noción preexistente de radio de Bohr. Usando dichas descomposiciones mostramos, para todo el espectro de parámetros involucrados, cuál es el comportamiento asintótico de este radio.

También gracias a dichas descomposiciones, conseguimos resultados acerca de los conjuntos de convergencia monomial de otras familias de funciones holomorfas. Para $H_b(\ell_r)$, las funciones enteras y acotadas en conjuntos acotados en ℓ_r, caracterizamos aquellos conjuntos de convergencia cuando $1 < r \leqslant 2$. Cuando $r > 2$ logramos hacerlo para $H_b(\ell_{r,\infty})$ y damos cotas superiores e inferiores en el caso de $H_b(\ell_r)$. Hacemos un avance significativo para el caso de funciones holomorfas y acotadas en la bola de ℓ_r con $1 < r \leqslant 2$.

Monomial decomposition and summability for holomorphic functions in high dimensions

Abstract

This thesis aims to contribute to the theory of holomorphic functions and homogeneous polynomials in several and infinitely many variables. We study several objects that, in one way or another, involve the summability of homogeneous polynomial coefficients depending on their uniform norm.

We compare the uniform and the coefficients norms in spaces of homogeneous polynomials in several complex variables. In particular, we study the asymptotic behaviour of the equivalence constants between these two norms when the number of variables goes to infinity.

We give a complete description of the asymptotic behaviour of the mixed unconditional constants in spaces of homogeneous polynomials with finite variables. To achieve this it is essential that we study the sets of monomial convergence for these spaces of polynomials. In this sense, we make a substantial progress in the characterization of these sets in the case of homogeneous polynomials on ℓ_r when $1 < r \leqslant 2$ proving an open conjecture in the area.

We introduce novel decompositions of the set of monomials, which are very useful to attack problems of unconditionality and summability allowing an adequate management of the dependence between the degree of homogeneity and the number of variables in certain inequalities.

We define the mixed Bohr radius extending the preexisting notion of Bohr radius. Using these decompositions we show, for the entire spectrum of parameters involved, the asymptotic behaviour of this radius.

Also thanks to these decompositions, we get results for the sets of monomial convergence of other families of holomorphic functions. For $H_b(\ell_r)$, the entire functions and bounded in their bounded sets in ℓ_r, we characterize their sets of convergence when $1 < r \leqslant 2$. When $r > 2$ we manage to do it for $H_b(\ell_{r,\infty})$ and give upper and lower bounds in the case of $H_b(\ell_r)$. We make significant progress in the case of holomorphic and bounded functions in the ball of ℓ_r with $1 < r \leqslant 2$.

Introduction

Complex analysis in one variable is one of the most influential theories of mathematics. Many other fields of science, inside and outside of mathematics are strongly related to it: number theory, differential equations, harmonic analysis, fluid mechanics, electromagnetism, quantum mechanics, among others, would not be same without the power of complex analysis. Within this theory, one of the fundamental cornerstones is the equivalence between differentiability and analyticity. That is, it is the same for a function to be complex differentiable in an open set of the complex plane as to be written locally as a power series. This fact still applies to complex functions of several variables, but... What happens in case there are infinitely many variables?

The idea of developing a theory of complex analysis in infinitely many variables starts at the beginning of the 20th century with the works of Hilbert, Fréchet and Gâteaux, among others. In this context, the problem of relating the differentiability of a function with its Taylor series expansion becomes more subtle. Studying functions in the ball of c_0, Hilbert proposes to work with those mappings that can be written as infinite linear combination of the monomials, that is

$$f(z) = \sum_{\alpha=(\alpha_1,\ldots,\alpha_n)\in\mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z_1^{\alpha_1}\cdots z_n^{\alpha_n}.$$

Hilbert [Hil09] declares, without a proof, that these should be exactly the holomorphic functions in the ball of c_0. However, Toeplitz [Toe13] gives an example of a holomorphic function and a point in c_0 for which its monomial expansion does not converge, contradicting Hilbert's statement.

In the development of the adequate notion of holomorphy in infinitely many variables, the vision given by the Fréchet-differentiability of the function becomes strong. Given X a Banach space over $\mathbb{C}$ and an open set $U \subset X$, a complex valued function is holomorphic on U if it meets the differentiability condition given by the difference quotient. This point of view shows to be very fruitful allowing a development of a consistent and productive theory of infinite dimensional complex analysis. A fundamental fact of this definition is that, for every holomorphic function $f : U \subset X \to \mathbb{C}$ and every point $z_0 \in U$ it exists a sort of Taylor expansion: there are m-homogeneous polynomials $P_m(f)(z_0)$ such that

$$f(z) = f(z_0) + \sum_{m\geqslant 1} P_m(f)(z_0)(z - z_0),$$

for every z in a neighborhood of z_0. These homogeneous polynomials are somewhat more abstract objects than monomials but, in any case, very useful.

In some Banach spaces, for example in ℓ_∞ or c_0, it is possible to think of monomials of the form $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ with $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^{(\mathbb{N})}$. This is due to the fact that the elements of these spaces are sequences. This additional structure, which is not shared by all the Banach spaces, allows us to talk about a monomial expansion. The definition through differentiability provides a more general theory, while Hilbert's idea has the potential to be based on the more concrete concept of the monomials. Looking for a reconciliation between Hilbert's vision and the one used to define holomorphic functions, some questions emerge. Given a Banach space X (with an extra structure such that the monomials make sense) and a holomorphic function f in the ball of X, we have the following queries:

- Does it always make sense to think of an expression of form $\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z^\alpha$?
- If it does, for which elements z in the ball of X do we have $f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z^\alpha$?

These questions promote the study of the monomial convergence in families of holomorphic functions. A systematic investigation of these type of problems begins in [DMP09]. For those Banach spaces X that are also sequence spaces, that is, $X \subset \mathbb{C}^{\mathbb{N}}$ with continuous inclusions $\ell_1 \hookrightarrow X \hookrightarrow \ell_\infty$, we can always consider the notion of monomial. Given a Banach sequence space X every holomorphic function in the whole space X (or in its ball, for example) f defines a sequence $(a_\alpha(f))_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}$ such that

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z^\alpha,$$

for every z with finite support in the domain of f. Thus, given a family of holomorphic functions $\mathcal{F}$, its set of monomial convergence is defined as the largest set where the monomial expansion converges for every function in the family, i.e.,

$$mon\mathcal{F} = \left\{ z \in \mathbb{C}^{\mathbb{N}} : \sum_{\alpha} |a_\alpha(f) z^\alpha| < \infty \text{ for every } f \in \mathcal{F} \right\}.$$

In [DMP09] the authors introduce two key results in the theory linking the study of the monomial convergence sets of certain families of holomorphic functions with the asymptotic behavior of the mixed unconditionality constant of spaces of homogeneous polynomial. In addition, several results are obtained in order to characterize these sets for some natural families of holomorphic functions, laying the foundations of their study.

It is not simple at all to describe these sets of monomial convergence in general, not even in concrete examples of families of holomorphic functions. In the case of ℓ_1, thanks to the work of Ryan in [Rya87] and Lempert in [Lem99], we know that $mon\mathcal{P}(^m\ell_1) = \ell_1$ for all $m \in \mathbb{N}$ and $monH_\infty(B_{\ell_1}) = B_{\ell_1}$. For ℓ_∞ (the other end of the spectrum), in the outstanding work of Bayart, Defant, Frerick, Maestre and Sevilla-Peris [BDF$^+$17] they manage to prove $mon\mathcal{P}(^m\ell_\infty) = \ell_{\frac{2m}{m-1},\infty}$ for all $m \in \mathbb{N}$ and $B \subset monH_\infty(B_{\ell_\infty}) \subset \overline{B}$, with

$$B := \left\{ z \in B_{\ell_\infty} : \limsup_{n\to\infty} \frac{1}{\sqrt{\log(n)}} \left(\sum_{k=1}^{n} (z_k^*)^2 \right)^{1/2} < 1 \right\}.$$

In [BDS19] an accurate description of the monomial convergence set of $\mathcal{P}(^m\ell_{p,\infty})$, for $p > 2$, is given and this is the only other case where a precise description of the set of monomial convergence is known. In [BDS19] and [DMP09] the authors also show some upper and lower bounds of these sets for $H_\infty(B_{\ell_p})$.

While the foundations of infinite dimensional holomorphy were discussed, Harald Bohr finds a deep link between this flourishing theory and number theory. He devotes many efforts to the study of Dirichlet series. These series, under some conditions, define holomorphic functions on certain domains of the complex plane. A paradigmatic example of them is the Riemann zeta function. In 1913 in [Boh13] Bohr analyzes the distinct regions in which these types of functions converge in different senses. He makes great advances in the study of these issues, leaving a question unanswered: What is the thickness of the gap between the region of absolute convergence and the region of uniform convergence for these series? He builds a bridge between the Dirichlet series and the holomorphic functions in the ball of c_0, that would be essential to solve this question years later. This link is known nowadays as the Bohr transform and, thanks to the decomposition into prime numbers of the integers, it gives a biunivocal mapping from those Dirichlet series which are convergent and bounded in $\{z \in \mathbb{C} : \ Re(z) > 0\}$ into the set of holomorphic and bounded functions in the ball of c_0.

Using Taylor's decomposition into homogeneous polynomials for holomorphic functions in B_{ℓ_∞}, in 1931 Bohnenblust and Hille go one step further in the research that Bohr had initiated. Inspired by Littlelwood's 4/3 inequality, they are able to translate the problem posed by Bohr into the study of certain inequalities that relate the summability of the coefficients of homogeneous polynomials with their uniform norm in the ball of ℓ_∞. In [BH31] they prove that for any m-homogeneous polynomial in n complex variables $P = \sum_{\alpha_1+\cdots+\alpha_n=m} a_\alpha(P)z^\alpha$, it holds

$$\left(\sum_{\alpha_1+\cdots+\alpha_n=m} |a_\alpha(P)|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leqslant C_m \sup_{z\in B_{\ell_\infty^n}} |P(z)|, \tag{1}$$

with $C_m > 0$ a constant independent of the number of variables n. This result allows them to determine the value of the gap between regions of absolute and uniform convergence for Dirichlet series.

Motivated by these questions, in 1914, Bohr raises another problem about holomorphic functions in one variable. In [Boh14] he proposes to find the largest radius $r > 0$ such that for every holomorphic function on the unit disk of the complex plane given by

$f(z) = \sum_{k\geqslant 0} a_k(f)z^k$ it holds

$$\sup_{|z|<r} \sum_{k\geqslant 0} |a_k(f)z^k| \leqslant \sup_{|z|<1} \left| \sum_{k\geqslant 0} a_k(f)z^k \right|. \tag{2}$$

Bohr solves this problem by showing that the maximal radius is 1/3.

In 1989, many years later, this concept is resumed in [DT89], generalizing and connecting it with notions of the local theory of Banach space. It begins a systematic study of what is known as the n-dimensional Bohr radius. Given some norm over $\mathbb{C}^n$, that is $X = (\mathbb{C}^n, \|\cdot\|_X)$, it is studied the variant of Bohr's problem that comes from replacing the unit disk by the ball of X. Thus, the problem is to seek for the largest radius $r > 0$ such that for any holomorphic function in the ball of X given by $f(z) = \sum_{\alpha\in\mathbb{N}_0^n} a_\alpha(f)z^\alpha$ it follows

$$\sup_{\|z\|_X<r} \sum_{\alpha\in\mathbb{N}_0^n} |a_\alpha(f)z^\alpha| \leqslant \sup_{\|z\|_X<1} \left| \sum_{\alpha\in\mathbb{N}_0^n} a_\alpha(f)z^\alpha \right|.$$

This maximal radius is called the Bohr radius of the ball of X and it is denoted by $K(B_X)$. In this way we can say that Bohr's result is translated as $K(\mathbb{D}) = 1/3$. It is worth mentioning that for the ball of no space with a dimension larger than one the precise value of Bohr's radius is known. The advances in this field are linked to understanding the asymptotic behavior of this radius for the balls of certain spaces when its dimension tends to infinity. Many efforts have been devoted to understanding the asymptotic behavior of $K(B_{\ell_p^n})$, particularly in the articles [DT89, KB97, Aiz00, Boa00, BB04, Bay12, DP06] where the problem is treated in one way or another, achieving partial advances.

In one of the most influential articles of the area [DFOC$^+$11] a fundamental step is taken, the authors manage to prove that Bohr's radius of the n-dimensional polydisk has the following asymptotic behaviour

$$K(\mathbb{D}^n) \sim \sqrt{\frac{\log(n)}{n}}.$$

This is achieved through a deep study of the inequality achieved by Bohnenblust and Hille given in (1) (see also [BPSS14]). In [DF11] the authors manage to describe the asymptotic behavior of Bohr's radius for the ball of ℓ_p^n for all $1 \leqslant p \leqslant \infty$ obtaining that

$$K(B_{\ell_p^n}) \sim \left(\frac{\log(n)}{n}\right)^{1-\frac{1}{\min(2,p)}}.$$

Many of these investigations relate Bohr's radius of the ball of a certain Banach space X (with finite dimension) to the unconditionality constants in spaces of homogeneous polynomials over X.

The purpose of this thesis is to resume several of the ideas that arise from understanding Hilbert's ideas (describing holomorphic functions in infinite dimension), Harald Bohr's studies (that gave way to research around the radius that bears his name) and the coefficient summability of homogeneous polynomials (as it appears in the inequality studied by

Bohnenblust and Hille). It is addressed the intimate relationship between all these concepts with the unconditionality in spaces of homogeneous polynomials which is, in some way, the idea that links them and is present throughout the thesis.

All this, in one way or another, involves the summability of the coefficients of homogeneous polynomials depending on their uniform norm. For this reason we begin by comparing the uniform and coefficient norms in spaces of homogeneous polynomials in several complex variables. In particular, we study the asymptotic behavior of the equivalence constants between these two norms when the number of variables tends to infinity.

We introduce novel decompositions of the monomials, which are very useful to attack problems of unconditionality and summability. These decompositions allow an adequate management of the dependence between the degree of homogeneity and the number of variables in certain inequalities. Thanks to this, we managed to give good descriptions of the set of monomial convergence for several natural families of holomorphic functions and successfully describe the asymptotic behavior of a generalization of the Bohr radius for the whole spectrum of parameters involved. We use some of these results to give a finished description of the asymptotic behavior of the mixed unconditionality constants for spaces of homogeneous polynomials.

Below we give a description of each chapter of the thesis.

Chapter 1 contains the notation and previous results of the general theories necessary for the presentation of the following chapters.

In Chapter 2 some variants of the Bohnenblust-Hille inequality (1) are developed. A general problem concerning the sumability of coefficients of m-homogeneous polynomials in n complex variables is studied. We seek to understand, for $1 \leqslant q, p \leqslant \infty$ fixed, the asymptotic behavior of the smaller constants $A^m_{p,q}(n)$ and $B^m_{q,p}(n)$ such that for every m-homogeneous polynomial in n complex variables $P = \sum_{\alpha_1+\cdots+\alpha_n=m} a_\alpha(P) z^\alpha$ it holds,

$$\left(\sum_{\alpha_1+\cdots+\alpha_n=m} |a_\alpha(P)|^q \right)^{1/q} \leqslant A^m_{p,q}(n) \sup_{z \in B_{\ell_p^n}} |P(z)|,$$

$$\sup_{z \in B_{\ell_p^n}} |P(z)| \leqslant B^m_{q,p}(n) \left(\sum_{\alpha_1+\cdots+\alpha_n=m} |a_\alpha(P)|^q \right)^{1/q}.$$

Also, other variants of this type of inequalities are presented and the random polynomials are introduced (necessary throughout the thesis). Finally, some results obtained on the behavior of $A^m_{p,q}(n)$ and $B^m_{q,p}(n)$ are applied, obtaining conclusions about complex interpolation in spaces of homogeneous polynomials and von Neumann inequalities in operator theory.

In Chapter 3 the sets of monomial convergence for families of holomorphic functions are defined. Section 3.3 presents families of holomorphic functions with the rearrangement property, an indispensable technical attribute in all descriptions for sets monomial convergence. It is proven that the most natural and studied families have this property. Finally it is proven that, for $1 < r \leqslant 2$, $m \geqslant 2$ and $q = (mr')'$ we have

$$\ell_q \subset mon\mathcal{P}(^m\ell_r), \tag{3}$$

proving a conjecture that appears in [DMP09].

Chapter 4 introduces the notion of mixed unconditionality in spaces of polynomials. It is shown that, in a precise sense, the mixed unconditionality in $\mathcal{P}(^m\mathbb{C}^n)$ does not depend on the base. Some results that link it with the sets of monomial convergence for $H_\infty(B_X)$ and $\mathcal{P}(^mX)$ are presented. Thanks to these, the inclusion obtained for $mon\mathcal{P}(^m\ell_r)$ in (3), and the results on the constants $A^m_{p,q}(n)$ and $B^m_{q,p}(n)$, a description of the correct asymptotic behavior of the mixed unconditionality constant for the entire range of values on which it depends is achieved.

In Chapter 5 the Bohr radius is discussed. The mixed Bohr radius, that generalizes the classic Bohr radius, is presented and it is shown, imitating what happens for the classic case, that it is related to the mixed unconditionality in spaces of polynomials. The asymptotic behavior of the mixed Bohr radius for the entire range of values on which it depends is characterized. For this, tools that come from the study of sets of monomial convergence are used. A novel decomposition of the monomials is introduced: the *bounded decomposition*. This decomposition is based on distinguishing monomials in terms of the maximum degree of their variables. This allows to handle certain inequalities with the technical subtlety necessary to achieve the correct asymptotic order.

In Chapter 6, the sets of monomial convergence of the spaces $H_b(\ell_r)$ for $1 < r \leqslant 2$ and $H_b(\ell_{r,s})$ for $2 \leqslant r, s \leqslant \infty$ are studied. The second monomial decomposition is presented: the *factorization decomposition*. Unlike other decompositions that appear in the literature, which consist on a partition of the set of monomials, this technique allows factoring each multi-index into multi-indexes with a very specific structure. Being more precise, it consists in factoring each monomial as the product of a tetrahedral monomial and another monomial in which every variable is raised to an even power. From it the following characterizations are achieved:

- For $1 < r \leqslant 2$,

$$monH_b(\ell_r) = \left\{ z \in \mathbb{C}^{\mathbb{N}} : \sup_{n \geqslant 1} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-\frac{1}{r}}} < \infty \right\}.$$

- For $2 < r \leqslant \infty$,

$$monH_b(\ell_{r,\infty}) = \left\{ z \in \mathbb{C}^{\mathbb{N}} : \sup_{n \geqslant 1} \frac{\left(\sum_{k=1}^n k^{\frac{2}{r}} (z_k^*)^2\right)^{1/2}}{\sqrt{\log(n+1)}} < \infty \right\}.$$

Also there are given quiet tight upper and lower bounds for $monH_b(\ell_{r,s})$ with $2 \leqslant r \leqslant \infty$ and $2 < s < \infty$.

In Chapter 7 we use the results obtained in Chapter 6 to provide descriptions of the sets of monomial convergence for $H_\infty(B_{\ell_r})$ when $1 < r \leqslant 2$. This descriptions improve those known so far, characterizing the geometry of this sets in a very specific sense that is developed in the chapter.

Chapter 8 is the last one, here it is resumed the result of Chapter 3 written in (3). Thanks to interpolation techniques in cones, a substantial improvement is achieved, obtaining for $1 < r \leqslant 2$, $m \geqslant 5$ and $q := (mr')'$ that

$$\ell_{q,\frac{m}{\log(m)}} \subset mon\mathcal{P}(^m\ell_r) \subset \ell_{q,\infty}.$$

Contents

Chapter 1

Preliminaries

In this chapter we set the notation for the rest of this monograph. Here we give the background notions of polynomials and holomorphic functions over Banach spaces. We introduce some of the algebraic, topologic and geometric structures that will be key to understand the rest of the chapters.

As usual $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the sets of natural, integer, real and complex numbers respectively. For the open unit disc in $\mathbb{C}$ we use $\mathbb{D}$ and for the set of unimodular elements of $\mathbb{D}$, usually called the unidimensional torus, $\mathbb{T}$.

Given $k \in \mathbb{N}$ we denote by S_k to the group of all the permutations of $\{1, \ldots, k\}$, i.e., the biyective mappings $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$. For the group of permutations of $\mathbb{N}$ we use $S_{\mathbb{N}}$.

1.1 Banach Spaces

We will use X for a general Banach space and $\| \cdot \|_X$ to refer to its norm, we write $B_X := \{x \in X : \; \|x\|_X < 1\}$ to denote its ball. To name the ball of radius $r > 0$ centered at $x \in X$ we will use $B_r(x)$. Throughout the text all the Banach spaces will be over the complex field $\mathbb{C}$. Recall a subset $U \subset X$ is said to be balanced if, for every $\lambda \in \overline{\mathbb{D}}$ the set $\lambda \cdot U := \{\lambda x : x \in U\}$ is contained in U. Given X and Y Banach spaces we will denote the space of linear and bounded operators from X to Y by $\mathcal{L}(X, Y)$, and given $T \in \mathcal{L}(X, Y)$ we will use $\|T\|_{\mathcal{L}(X,Y)} := \sup_{x \in B_X} \|T(x)\|_Y$ or just $\|T\|$ to lighten the notation when the context allows the vagueness. We will use $X' = \mathcal{L}(X, \mathbb{C})$ to denote the dual space of X.

1.1.1 Basis and Unconditionality

A sequence $(b_n)_{n \geqslant 1} \subset X$ is said to be a *basis* for X if, for every $x \in X$ there is a scalar sequence $(a_n(x))_{n \geqslant 1} \subset \mathbb{C}$ such that $x = \lim_{N \to \infty} \sum_{n=1}^{N} a_n(x) b_n$. We can consider for a basis

$(b_n)_{n\geqslant 1}$ the *biorthogonal mappings*

$$\begin{aligned} b_n^* : X &\to \mathbb{C} \\ x = \sum_{k\geqslant 1} a_k(x) b_k &\mapsto a_n(x). \end{aligned}$$

Whenever b_n^* is a continuous functional for every $n \in \mathbb{N}$ the sequenece $(b_n)_{n\geqslant 1}$ is said to be a *Schauder basis*. The sequence $(b_n^*)_{n\geqslant 1}$ is called the *dual basis* of $(b_n)_{n\geqslant 1}$ and it is a *basic sequence*, this means it is a Schauder basis of the subspace of X' generated by $\overline{[b_n^* : n \in \mathbb{N}]} \subset X'$. In the case of reflexive spaces, the dual basis is always a basis for the dual space.

Given $1 \leqslant p < \infty$, the well known Banach space

$$\ell_p := \left\{ z = (z_1, \ldots, z_n, \ldots) \in \mathbb{C}^{\mathbb{N}} : \sum_{n\geqslant 1} |z_n|^p < \infty \right\},$$

endowed with the norm $\|z\|_{\ell_p} := \left(\sum_{n\geqslant 1} |z_n|^p\right)^{1/p}$ is an example of a space with Schauder basis. Here the canonical vectors

$$e_n := (0, \ldots, 0, \underbrace{1}_{n^{\text{th}} \text{ position}}, 0, \ldots) \text{ for every } n \in \mathbb{N},$$

form the Schauder basis $(e_n)_{n\geqslant 1}$. Notice that

$$\ell_\infty := \left\{ z = (z_1, \ldots, z_n, \ldots) \in \mathbb{C}^{\mathbb{N}} : \sup_{n\geqslant 1} |z_n| < \infty \right\},$$

with the usual norm $\|z\|_\infty = \sup_{n\geqslant 1} |z_n|$ does not support a Schauder basis, as it is not separable. On the other hand the canonincal vectors form a Schauder basis for the closed subspace

$$c_0 := \left\{ z \in \ell_\infty : \lim_{n\to\infty} z_n = 0 \right\}.$$

For $1 \leqslant p \leqslant \infty$ we write p' for its *conjugate exponent* (i.e., $\frac{1}{p} + \frac{1}{p'} = 1$). Recall that for $1 < p < \infty$ the dual space for ℓ_p is $\ell_{p'}$ and also $c_0' = \ell_1$ and $\ell_1' = \ell_\infty$. The space ℓ_1 is an example of Banach spaces with Shcauder basis for which its dual does not support one.

Another very studied concept in geometry of Banach spaces is its unconditionality. This concept will be one of the key elements of this thesis. Given a sequence $(x_n)_{n\geqslant 1}$ in a Banach space X, the series $\sum_{n\geqslant 1} x_n$ is said to *converge unconditionally* if $\sum_{n\geqslant 1} x_{\sigma(n)}$, for every permutation σ of $\mathbb{N}$. There are plenty of equivalent definitions for this concept which can be found as the *Omnibus Theorem on Unconditionality Summability* on [DJT95, Theorem 1.9], here we need only a few of those equivalences.

Theorem 1.1.1. *For a sequence $(x_n)_{n\geqslant 1}$ in a Banach space X they are equivalent*

(i) The series $\sum_{n\geqslant 1} x_n$ converges unconditionally.

(ii) The series $\sum_{n\geqslant 1} \varepsilon_n x_n$ *converges for every* $(\varepsilon_n)_{n\geqslant 1} \subset \mathbb{T}^{\mathbb{N}}$.

(iii) The series $\sum_{n\geqslant 1} \varepsilon_n x_n$ *converges for every* $(\varepsilon_n)_{n\geqslant 1} \in B_{\ell_\infty}$.

Given X a Banach spaces with Schauder basis $(x_n)_{n\geqslant 1}$ we say it is an *unconditional basis* for X whenever, each convergent series of the form $\sum_{n\geqslant 1} a_n x_n$ converges unconditionally. This means the convergence of the series representation of every element of the space does not depend on the order of the terms in the sum, which is a very good and useful qualitative property. Notice, for example, every orthonormal basis in a Hilbert space is an unconditional basis. In particular this holds for the Fourier basis of $L_2[0,1]$, with the practical benefit that any signal may be recovered by its harmonic components independently of the order of addition. Maybe that example illustrates the importance of this notion in many branches of analysis. The following theorem gives a more quantitative way to understand unconditionality (see [AK06, Theorem 3.1.3]).

Theorem 1.1.2. *Given* X *a Banach space with Schauder basis* $(x_n)_{n\geqslant 1}$ *the following are equivalent.*

(i) $(x_n)_{n\geqslant 1}$ *is an unconditional basis for* X.

(ii) There exists some constant $K > 0$ *such that for every* $(a_n)_{n\geqslant 1} \subset \mathbb{C}$ *and* $(\varepsilon_n)_{n\geqslant 1} \subset \mathbb{T}$ *it holds*

$$\left\| \sum_{n\geqslant 1} \varepsilon_n a_n x_n \right\|_X \leqslant K \left\| \sum_{n\geqslant 1} a_n x_n \right\|_X . \tag{1.1}$$

Given an unconditional basis $(x_n)_{n\geqslant 1}$ for the Banach space X we say it is a K-unconditional basis if it fulfills inequality (1.1). The optimal constant for that basis in (1.1) is denoted by $\chi((x_n)_{n\geqslant 1}; X)$.

The unconditional constant of the space X is given by

$$\chi(X) := \inf \chi((x_n)_{n\geqslant 1}; X),$$

where the infimum is taken over all the possible unconditional basis $(x_n)_{n\geqslant 1}$ of X.

Observe that the canonical vectors form a 1-unconditional basis for ℓ_p with $1 \leqslant p < \infty$ and c_0 as well as every orthonormal basis for a separable Hilbert space.

For finite dimensional Banach spaces every basis is unconditional as convergence is trivially guaranteed. Anyway, Theorem 1.1.2 allows to give a meaningful sense to the study of unconditionality in the finite dimensional case. It will be extremely useful to understand unconditional basis constants for finite dimensional spaces to tackle down problems on infinite dimensional ones.

1.1.2 Banach sequence spaces

In this section we introduce a special kind of Banach spaces that will be essential in our study of holomorphic functions thanks to its nice notion of coordinates.

For every pair of elements $x, y \in \mathbb{C}^{\mathbb{N}}$ we will use

$$x \cdot y := (x_1 y_1, x_2 y_2, \ldots, x_n y_n, \ldots)$$

to denote the coordinatewise product, and $|x|$ denotes the sequence $(|x_1|, |x_2|, \ldots, |x_n|, \ldots)$. If $|x_i| \leqslant |y_i|$ for every $i \in \mathbb{N}$ we write $|x| \leqslant |y|$. A *Banach sequence space* is a Banach space $(X, \|\cdot\|_X)$ such that $\ell_1 \subset X \subset \ell_\infty$ satisfying that, if $x \in \mathbb{C}^{\mathbb{N}}$ and $y \in X$ with $|x| \leqslant |y|$, then $x \in X$ and $\|x\|_X \leqslant \|y\|_X$. That is, if an element is bounded in norm by another that belongs to the space, first one must be in the space as well.

A non-empty open set $\mathcal{R} \subset X$ is called a *Reinhardt domain* if given $x \in \mathbb{C}^{\mathbb{N}}$ and $y \in \mathcal{R}$ such that $|x| \leqslant |y|$ then $x \in \mathcal{R}$. Given a bounded sequence x its *decreasing rearrangement* x^* is the sequence defined as

$$x_n^* = \inf\{\sup_{j \in \mathbb{N}\setminus J} |x_j| : J \subset \mathbb{N}, card(J) < n\}.$$

A Banach sequence space $(X, \|\cdot\|_X)$ is said to be symmetric if $x^* \in X$ if and only if $x \in X$ and, moreover $\|x\|_X = \|x^*\|_X$. A set $A \subset X$ is symmetric if $x \in A$ if and only if $x^* \in A$.

Proposition 1.1.3. *For every $x \in c_0$ there is some injective mapping $\sigma : \mathbb{N} \to \mathbb{N}$ such that $x_n^* = |x_{\sigma(n)}|$ for all $n \in \mathbb{N}$.*

We will say that a sequence $x \in \mathbb{C}^{\mathbb{N}}$ is decreasing whenever $|x|$ is decreasing.

The Banach spaces ℓ_p with $1 \leqslant p \leqslant \infty$ and c_0 are Banach sequence spaces. Also the more general Lorentz spaces are good examples of this spaces, let us recall their definition. For $1 \leqslant p, q \leqslant \infty$ the *Lorentz space* $\ell_{p,q}$ is defined as

$$\ell_{p,q} := \left\{ z \in \mathbb{C}^{\mathbb{N}} : \|(k^{\frac{1}{p}-\frac{1}{q}} z_k^*)_{k \geqslant 1}\|_{\ell_q} < \infty \right\}.$$

Given $1 \leqslant p, q \leqslant \infty$, for $z \in \ell_{p,q}$ we define

$$\rho_{p,q}(z) := \|(k^{\frac{1}{p}-\frac{1}{q}} z_k^*)_{k \geqslant 1}\|_{\ell_q} = \begin{cases} \sup\limits_{k \geqslant 1} k^{\frac{1}{p}} |z_k^*| & \text{for } q = \infty, \\ \left(\sum\limits_{k \geqslant 1} k^{-1} |z_k^*|^q \right)^{\frac{1}{q}} & \text{for } q < \infty, p = \infty, \\ \left(\sum\limits_{k \geqslant 1} k^{\frac{q}{p}-1} |z_k^*|^q \right)^{\frac{1}{q}} & \text{for } q, p < \infty. \end{cases}$$

Whenever $1 \leqslant q \leqslant p \leqslant \infty$, $\|z\|_{\ell_{p,q}} := \rho_{p,q}(z)$ defines a norm over $\ell_{p,q}$ that makes this spaces a Banach sequence space. Observe that $\ell_{p,p} = \ell_p$ for every $1 \leqslant p \leqslant \infty$. For $1 \leqslant p \leqslant q \leqslant \infty$ the mapping $\rho_{p,q}$ does not fulfill the triangle inequality but it is a complete quasi-norm in $\ell_{p,q}$, i.e. there is $c > 0$ such that for every $z, w \in \ell_{p,q}$ it holds

$$\rho_{p,q}(z + w) \leqslant c\,(\rho_{p,q}(z) + \rho_{p,q}(w)).$$

We will denote in general $\|\cdot\|_{\ell_{p,q}} = \rho_{p,q}(\cdot)$ even in the cases this is not a norm. This problem can be fixed by considering these spaces endowed with the norm given, for $z \in \ell_{p.q}$, by

$$\|z\|_{\ell_{(p,q)}} = \left(\sum_{n=1}^{\infty} n^{\frac{q}{p}-1} \left(\frac{1}{n} \sum_{k=1}^{n} z_k^* \right)^q \right)^{1/q}.$$

For $1 < p, q \leqslant \infty$ and $z \in \ell_{p,q}$, it holds

$$\rho_{p,q}(z) \leqslant \|z\|_{\ell_{(p,q)}} \leqslant p' \rho_{p,q}(z), \tag{1.2}$$

so we can always work with the quasi-norm $\rho_{p,q}(\cdot)$ and treat $(\ell_{p,q}, \rho_{p,q}(\cdot))$ as a Banach sequence space if we are willing to pay p' as a price every time we do so (see [BS88, Lemma 4.5]). Also we need a result concerning the dual spaces of Lorentz spaces, which is an adaptation to the case we need of [BS88, Corollary 4.8].

Theorem 1.1.4. *For $1 < p < \infty$ and $1 \leqslant q < \infty$ the dual space $(\ell_{p,q})'$ is isomorphic to $\ell_{p',q'}$.*

One last particular family of Banach sequence spaces we will need are the Marcinkiewicz spaces. Let $\Psi = (\Psi(n))_{n=0}^{\infty}$ be an increasing sequence of nonnegative real numbers with $\Psi(0) = 0$ and $\Psi(n) > 0$ for every $n \in \mathbb{N}$. These functions are usually known as symbols. The *Marcinkiewicz sequence space* associated to the symbol Ψ, denoted by m_Ψ, is the vector space of all bounded sequences $(z_n)_n$ such that

$$\|z\|_{m_\Psi} := \sup_{n \geqslant 1} \frac{\sum_{k=1}^{n} z_k^*}{\Psi(n)} < \infty.$$

For X a Banach sequence space and n a natural number we consider the n^{th} projection

$$\begin{aligned} \pi_n : X &\to \mathbb{C}^n \\ (x_1, \ldots, x_n, \ldots) &\mapsto (x_1, \ldots, x_n), \end{aligned} \tag{1.3}$$

and the n^{th} inclusion

$$\begin{aligned} \iota_n : \mathbb{C}^n &\to X \\ (x_1, \ldots, x_n) &\mapsto (x_1, \ldots, x_n, 0, \ldots). \end{aligned} \tag{1.4}$$

We denote X_n to $\mathbb{C}^n$ endowed with the quotient norm induced by ι_n, i.e.

$$\|(z_1, \ldots, z_n)\|_{X_n} := \inf\{\|x\|_X : \iota_n(x) = (z_1, \ldots, z_n)\}.$$

Observe that this construction makes $\pi_n : X \to X_n$ and $\iota_n : X_n \to X$ norm one operators. Sometimes it will useful to identify X_n with $i_n(X_n) \subset X$. A very important example is $\ell_p^n := (\ell_p)_n$, this is the Banach space of all n-tuples $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ endowed with the norm $\|(z_1, \ldots, z_n)\|_p = \left(\sum_{i=1}^{n} |z_i|^p \right)^{1/p}$ if $1 \leqslant p < \infty$, and $\|(z_1, \ldots, z_n)\|_\infty = \max_{i=1,\ldots,n} |z_i|$ for $p = \infty$. We will also consider $\ell_{p,q}^n := (\ell_{p,q})_n$ for $1 \leqslant p, q \leqslant \infty$.

1.2 Homogeneous polynomials and multilinear maps.

Given $X_1, \dots X_m, Y$ Banach spaces a *multilinear map* (or *multilinear form*) from $X_1 \times \cdots \times X_m$ to Y is a function $T : X_1 \times \cdots \times X_m \to Y$ which is linear in every coordinate. We also name a function like that an m-linear map, being more specific. The set of all m-linear map from $X_1 \times \cdots \times X_m$ to Y is a vector space with the usual sum and scalar product of functions inherited from Y. Recall that a multilinear map $T : X_1 \times \cdots \times X_m \to Y$ is bounded or continuous whenever

$$\|T\|_{\mathcal{L}(X_1,\dots,X_n;Y)} := \sup\{\|T(x_1, \dots, x_m)\|_Y : x_1 \in B_{X_1}, \dots, x_m \in B_{X_m}\} < \infty,$$

and the set of all bounded m-linear maps from $X_1 \times \cdots \times X_m$ to Y equiped with the norm $\|\cdot\|_{\mathcal{L}(X_1,\dots,X_m;Y)}$ is a Banach space which we denote by $\mathcal{L}(X_1, \dots, X_m; Y)$. Whenever $Y = \mathbb{C}$ we simply use $\mathcal{L}(X_1, \dots, X_m)$, and if $X = X_1 = \cdots X_m$ we use $\mathcal{L}(^m X; Y)$.

Let $W_1, \dots W_m, X_1 \dots, X_m$ be Banach spaces and $u_i \in \mathcal{L}(W_i, X_i)$ let us consider the mapping

$$\begin{aligned}(u_1, \dots, u_m) : W_1 \times \cdots \times W_m &\to X_1 \times \cdots \times X_m \\ (w_1, \dots, w_m) &\mapsto (u_1(w_1), \dots, u_m(w_m)).\end{aligned}$$

Proposition 1.2.1. *Let $W_1, \dots W_m, X_1 \dots, X_m, Y, Z$ Banach spaces. For $u_i \in \mathcal{L}(W_i, X_i)$ for every $1 \leqslant i \leqslant m$, $v \in \mathcal{L}(Y, Z)$ and $T \in \mathcal{L}(X_1, \dots, X_m; Y)$ it holds $v \circ T \circ (u_1, \dots, u_m) \in \mathcal{L}(W_1, \dots W_m; Z)$ and*

$$\begin{aligned}\|v \circ T \circ (u_1, \dots, u_m)\|_{\mathcal{L}(W_1,\dots W_m;Z)} \leqslant \\ \leqslant \|v\|_{\mathcal{L}(Y,Z)} \|T\|_{\mathcal{L}(X_1,\dots,X_m;Y)} \|u_1\|_{\mathcal{L}(W_1,X_1)} \cdots \|u_m\|_{\mathcal{L}(W_m,X_m)}.\end{aligned}$$

Given a pair of Banach spaces X, Y and $T \in \mathcal{L}(^m X; Y)$ a mapping

$$\begin{aligned}P : X &\to Y \\ x &\mapsto T(x, \dots, x),\end{aligned}$$

is an *m-homogeneous polynomial* from X to Y. We say P is bounded or continuous if

$$\|P\|_{\mathcal{P}(^m X;Y)} := \sup_{x \in B_X} \|P(x)\|_Y < \infty.$$

In other words, is we consider the diagonal inclusion

$$\begin{aligned}\Delta_m : X &\to X^m \\ x &\mapsto (x, \dots, x),\end{aligned}$$

a bounded m-homogeneous polynomial is a map $P = T \circ \Delta_m$ where $T \in \mathcal{L}(^m X; Y)$.

Notice that for every m-homogeneous polynomial P defined by $T \in \mathcal{L}(^m X; Y)$ it holds

$$\|P\|_{\mathcal{P}(^m X;Y)} \leqslant \|T\|_{\mathcal{L}(^m X;Y)}. \tag{1.5}$$

The set of all m-homogeneous bounded polynomials from X to Y with the norm $\|\cdot\|_{\mathcal{P}(^mX;Y)}$ is a Banach space that we denote $\mathcal{P}(^mX;Y)$. Whenever $Y = \mathbb{C}$ we directly use $\mathcal{P}(^mX)$ instead of $\mathcal{P}(^mX;\mathbb{C})$. For $P \in \mathcal{P}(^mX;Y)$ there is always a multilinear map $T \in \mathcal{L}(^mX;Y)$ such that $P = T \circ \Delta_m$, we say T is a m-linear map associated to P.

For a general map $g : U \to Y$ with Y a Banach space and some subset $V \subset U$ a useful notation will be

$$\|g\|_V := \sup_{v \in V} \|g(v)\|_Y.$$

Let $P \in \mathcal{P}(^mX;Y)$ observe that with this notation $\|P\|_{B_X} = \|P\|_{\mathcal{P}(^mX;Y)}$. A simple but central property of m-homogeneous polynomials is, as their name suggest, its homogeneity. This is, given $\lambda \in \mathbb{C}$, $x \in X$ and $P \in \mathcal{P}(^mX;Y)$, it holds

$$P(\lambda x) = \lambda^m P(x).$$

The homogeneous polynomials over Banach spaces have the nice and very important ideal property.

Proposition 1.2.2. *Let W, X, Y, Z Banach spaces. For $u \in \mathcal{L}(W,X)$, $v \in \mathcal{L}(Y,Z)$ and $P \in \mathcal{P}(^mX;Y)$ it holds $v \circ P \circ u \in \mathcal{P}(^mW;Z)$ and*

$$\|v \circ P \circ u\|_{\mathcal{P}(^mW;Z)} \leqslant \|v\|_{\mathcal{L}(Y,Z)} \|P\|_{\mathcal{P}(^mX;Y)} \|u\|^m_{\mathcal{L}(W,X)}.$$

In particular, for Banach sequence spaces we have the following extremely useful corollary.

Corollary 1.2.3. *Let X be a Banach sequence space and $P \in \mathcal{P}(^mX)$. For every $n \in \mathbb{N}$ it holds $P_n := P \circ \pi_n \in \mathcal{P}(^mX_n)$ and $\|P_n\|_{\mathcal{P}(^mX_n)} \leqslant \|P\|_{\mathcal{P}(^mX)}$.*

With this in mind, even though our main interest will be placed in the spaces of continuous polynomial on infinite dimensional spaces, it will be very useful to study their finite dimensional versions.

An m-homogeneous polynomial in n complex variables is a function $P : \mathbb{C}^n \to \mathbb{C}$ of the form

$$P(z_1, \dots, z_n) = \sum_{\alpha \in \Lambda(m,n)} a_\alpha z^\alpha,$$

where $\Lambda(m,n) := \{\alpha \in \mathbb{N}_0^n : |\alpha| := \alpha_1 + \cdots + \alpha_n = m\}$, $z^\alpha := z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ and $a_\alpha \in \mathbb{C}$.

Another way of writing a polynomial P is as follows:

$$P(z_1, \dots, z_n) = \sum_{\mathbf{j} \in \mathcal{J}(m,n)} c_\mathbf{j} z_\mathbf{j},$$

where $\mathcal{J}(m,n) := \{\mathbf{j} = (j_1, \dots, j_m) \in \mathbb{N}^m : 1 \leqslant j_1 \leqslant \cdots \leqslant j_m \leqslant n\}$, $z_\mathbf{j} := z_{j_1} \cdots z_{j_m}$ and $c_\mathbf{j} \in \mathbb{C}$. This two ways of indexing the coefficients of an homogeneous polynomial are related by the mapping

$$\begin{aligned} F : \Lambda(m,n) &\to \mathcal{J}(m,n) \\ \alpha = (\alpha_1, \dots, \alpha_n) &\mapsto \mathbf{j} = (1, \overset{\alpha_1}{\dots}, 1, \dots, n, \overset{\alpha_m}{\dots}, n). \end{aligned} \tag{1.6}$$

Note that $c_{\mathbf{j}} = a_\alpha$ with $\mathbf{j} = F(\alpha)$. Whenever $\mathbf{j} = F(\alpha)$ we may say $\alpha = \alpha(\mathbf{j})$ is associated to $\mathbf{j}$ and also $\mathbf{j} = \mathbf{j}(\alpha)$ is associated to α. We will use the fact that the cardinal of this index sets is

$$|\mathcal{J}(m,n)| = |\Lambda(m,n)| = \binom{m+n-1}{m}. \tag{1.7}$$

We refer to the elements $(z^\alpha)_{\alpha\in\Lambda(m,n)}$ (equivalently, $(z_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$) as the monomials. Notice that every $P \in \mathcal{P}(^m\mathbb{C}^n)$ defines a unique sequence of coefficients that can be written in two ways as $(a_\alpha(P))_{\alpha\in\Lambda(m,n)}$ and $(c_{\mathbf{j}}(P))_{\mathbf{j}\in\mathcal{J}(m,n)}$ such that

$$P(z) = \sum_{\alpha\in\Lambda} a_\alpha(P) z^\alpha = \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}}(P) z_{\mathbf{j}}.$$

Observe that $a_\alpha(P) = c_{\mathbf{j}}(P)$ for $F(\alpha) = \mathbf{j}$.

Another very useful index set, especially exploiting the connection between polynomials and multilinear forms, will be $\{1, \ldots, n\}^m$ which we will denote by the abbreviation $\mathcal{M}(m,n)$. Given $\mathbf{j} \in \mathcal{J}(m,n)$ there might be many $\mathbf{i} = (i_1, \ldots, i_m) \in \mathcal{M}(m,n)$ such that for some permutation $\sigma \in S_m$ it holds $\mathbf{j} = \sigma(\mathbf{i}) := (i_{\sigma(1)}, \ldots, i_{\sigma(m)})$. For every $\mathbf{j} \in \mathcal{J}(m,n)$ we consider the equivalence class

$$[\mathbf{j}] := \{\mathbf{i} \in \mathcal{M}(m,n) : \mathbf{j} = \sigma(\mathbf{j}) \text{ with } \sigma \in S_m\},$$

and by $|[\mathbf{j}]|$ we denote the cardinal of $[\mathbf{j}]$. Notice that for any $\mathbf{i} \in [\mathbf{j}]$ with $\sigma \in S_m$ such that $\mathbf{j} = \sigma(\mathbf{i})$ we have $z_{\mathbf{j}} = z_{\mathbf{i}} = (z_{\sigma(1)}, \ldots, z_{\sigma(n)})_{\mathbf{j}}$. Given $\alpha \in \Lambda(m,n)$, its cardinal may be calculated as $|[\alpha]| := \frac{m!}{\alpha!}$ where $\alpha! := \alpha_1! \cdots \alpha_n!$. Notice that, if $F(\alpha) = \mathbf{j}$ it holds $|[\mathbf{j}]| = |[\alpha]|$.

In the study of some spaces of homogeneous polynomials over Banach sequence spaces of infinite dimension it will be very useful to consider the sets

$$\mathcal{J}(m) := \bigcup_{n\geqslant 1} \mathcal{J}(m,n) = \{\mathbf{j} = (j_1, \ldots, \mathbf{j}_m) \in \mathbb{N}^m : 1 \leqslant j_1 \leqslant \cdots \leqslant j_m\},$$

and

$$\Lambda(m) := \bigcup_{n\geqslant 1} \Lambda(m,n) = \left\{\alpha \in \mathbb{N}_0^{(\mathbb{N})} : |\alpha| = m\right\}.$$

The following is a description of homogeneous polynomials over Banach sequence spaces depending on its finite dimensional projections.

Remark 1.2.4. Let X be a Banach sequence space, every $P \in \mathcal{P}(^mX)$ defines two unique coefficient sequences $(a_\alpha(P))_{\alpha\in\Lambda(m)}$ and $(c_{\mathbf{j}}(P))_{\mathbf{j}\in\mathcal{J}(m)}$ such that for every $n \in \mathbb{N}$ and $z \in \mathbb{C}^n$ it holds

$$P_n(z) = \sum_{\alpha\in\Lambda(m,n)} a_\alpha(P) z^\alpha = \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}}(P) z_{\mathbf{j}}.$$

Proof. For X a Banach sequence space and $P \in \mathcal{P}(^mX)$, by Corollary 1.2.3, we may consider its finite dimensional projections $(P_n)_{n\geqslant 1}$. For each $n \in \mathbb{N}$ the polynomial P_n defines unique coefficient sequences $(a_\alpha(P_n))_{\Lambda(m,n)}$ and $(c_{\mathbf{j}}(P_n))_{\mathbf{j}\in\mathcal{J}(m,n)}$. Notice that given two positive integers $n < N$ it holds $P_N \circ \pi_n = P_n$, which implies $(a_\alpha(P_N))_{\Lambda(m,n)} = (a_\alpha(P_n))_{\Lambda(m,n)}$ and $(c_{\mathbf{j}}(P_N))_{\mathbf{j}\in\mathcal{J}(m,n)} = (c_{\mathbf{j}}(P_n))_{\mathbf{j}\in\mathcal{J}(m,n)}$ for every $n < N$. This tells the dependence on n of the coefficient sequences of P_n is illusory, thus we may write $(a_\alpha(P_n))_{\Lambda(m,n)} = (a_\alpha(P))_{\Lambda(m,n)}$ and $(c_{\mathbf{j}}(P_n))_{\mathbf{j}\in\mathcal{J}(m,n)} = (c_{\mathbf{j}}(P))_{\mathbf{j}\in\mathcal{J}(m,n)}$. Finally for any $n \in \mathbb{N}$ given $z \in \mathbb{C}^n$ we have

$$P_n(z) = \sum_{\alpha\in\Lambda(m,n)} a_\alpha(P) z^\alpha = \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}}(P) z_{\mathbf{j}},$$

as we wanted. □

1.2.1 Symmetric multilinear maps

The connection between polynomial over Banach spaces and multilinear maps is clearly very deep. Indeed, for every polynomial there must be a multilinear map behind but, there might be more than one defining the same polynomial. For example, the polynomial

$$\begin{aligned} P : \mathbb{C}^2 &\to \mathbb{C} \\ (z_1, z_2) &\mapsto z_1^2 + 2z_1z_2 + z_2^2, \end{aligned}$$

may be given by two different bilinear map $T_1, T_2 \in \mathcal{L}(^2\mathbb{C}^2)$ where

$$\begin{aligned} T_1((x_1, x_2), (y_1, y_2)) &= x_1y_1 + 2x_1y_2 + x_2y_2, \\ T_2((x_1, x_2), (y_1, y_2)) &= x_1y_1 + x_1y_2 + y_1x_2 + x_2y_2. \end{aligned}$$

We say a multilinear form $T \in \mathcal{L}(^mX; Y)$ is symmetric if, given any permutation $\sigma \in S_m$ and any m-tuple $(x_1, \ldots, x_m) \in X^m$ it holds

$$T_\sigma(x_1, \ldots, x_m) := T(x_{\sigma(1)}, \ldots, x_{\sigma(m)}) = T(x_1, \ldots, x_m).$$

Notice that for any $\sigma \in S_m$ we have

$$\|T_\sigma\|_{\mathcal{L}(^mX;Y)} = \|T\|_{\mathcal{L}(^mX;Y)}. \tag{1.8}$$

Given any $T \in \mathcal{L}(^mX; Y)$ we consider its symmetrization $T^s \in \mathcal{L}(^mX; Y)$ given by

$$\begin{aligned} T^s : X^m &\to Y \\ (x_1, \ldots, x_m) &\mapsto \frac{1}{m!} \sum_{\sigma\in S_m} T_\sigma(x_1, \ldots, x_m), \end{aligned}$$

which is again an m-linear map which is symmetric. In fact we may define the symmetrization operator

$$\begin{aligned} \pi_s : \mathcal{L}(^mX; Y) &\to \mathcal{L}(^mX; Y) \\ T &\mapsto T^s. \end{aligned}$$

For any symmetric map $T \in \mathcal{L}(^m X; Y)$, as $T = T_\sigma$ for every $\sigma \in S_m$ and $\text{card}(S_m) = m!$ we have

$$T^s = \frac{1}{m!} \sum_{\sigma \in S_m} T_\sigma = T.$$

Also given any $T \in \mathcal{L}(^m X; Y)$ and by equation (1.8), it holds

$$\begin{aligned}\|\pi_s(T)\|_{\mathcal{L}(^m X;Y)} &= \|\frac{1}{m!} \sum_{\sigma \in S_m} T_\sigma\|_{\mathcal{L}(^m X;Y)} \\ &\leqslant \frac{1}{m!} \sum_{\sigma \in S_m} \|T_\sigma\|_{\mathcal{L}(^m X;Y)} \\ &= \frac{1}{m!} \sum_{\sigma \in S_m} \|T\|_{\mathcal{L}(^m X;Y)} = \|T\|_{\mathcal{L}(^m X;Y)}.\end{aligned}$$

By the previous remarks it turns out π_s is a projection. We call the subspace given by the range of projection the space of symmetric bounded m-linear forms and will be denote it by $\mathcal{L}^s(^m X; Y) := \pi_s(\mathcal{L}(^m X; Y))$.

Notice that we may define the surjective operator

$$\begin{aligned}\widehat{\ }: \mathcal{L}^s(^m X; Y) &\to \mathcal{P}(^m X; Y) \\ T &\mapsto \widehat{T} := T \circ \Delta_m.\end{aligned}$$

Given $P \in \mathcal{P}(^m X; Y)$ the polarization formula gives a unique symmetric associated m-linear form in $\mathcal{L}(^m X; Y)$, defined for any $(x_1, \ldots, x_m) \in X^m$ by

$$\check{P}(x_1, \ldots, x_m) := \frac{1}{2^m m!} \sum_{i=1}^{m} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \cdots \varepsilon_m P(\varepsilon_1 x_1 + \cdots + \varepsilon_m x_m).$$

Remark 1.2.5. For $P \in \mathcal{P}(^m \mathbb{C}^n)$, given $\alpha \in \Lambda(m, n)$ and $F(\alpha) = \mathbf{j} \in \mathcal{J}(m.n)$, for every $\mathbf{i} = (i_1, \ldots, i_m) \in [\mathbf{j}]$ it holds

$$a_\alpha(P) = c_{\mathbf{j}}(P) = \check{P}(e_{i_1}, \ldots, e_{i_m})|[\mathbf{j}]|,$$

Proof. Given $\mathbf{j} \in \mathcal{J}(m.n)$ and $\mathbf{i} \in [\mathbf{j}]$ there is some $\sigma \in S_m$ such that $\mathbf{j} = \sigma(\mathbf{i})$. As $\check{P}$ is symmetric it follows

$$\check{P}(e_{j_1}, \ldots, e_{j_m}) = \check{P}_\sigma(e_{j_1}, \ldots, e_{j_m}) = \check{P}(e_{i_1}, \ldots, e_{i_m}). \tag{1.9}$$

Notice that $\mathcal{M}(m, n) = \bigcup_{\mathbf{j} \in \mathcal{J}(m,n)} [\mathbf{j}]$ is a partition. Given $z \in \mathbb{C}^n$ and using equation (1.9) we have

$$\begin{aligned}P(z) = \check{P}(z, \ldots, z) &= \sum_{\mathbf{i} \in \mathcal{M}(m,n)} \check{P}(e_{i_1}, \ldots, e_{i_m}) z_{\mathbf{i}} \\ &= \sum_{\mathbf{j} \in \mathcal{J}(m,n)} \sum_{\mathbf{i} \in [\mathbf{j}]} \check{P}(e_{j_1}, \ldots, e_{j_m}) z_{\mathbf{j}} = \sum_{\mathbf{j} \in \mathcal{J}(m,n)} |[\mathbf{j}]| \check{P}(e_{j_1}, \ldots, e_{j_m}) z_{\mathbf{j}}.\end{aligned}$$

As there is a unique sequence of coefficients for P we have what we wanted to prove. ☐

The formula in (1.2.1) defines the linear operator

$$\check{}\colon \mathcal{P}(^mX;Y) \to \mathcal{L}^s(^mX;Y)$$
$$P \mapsto \check{P}.$$

The following theorem states that both operators $\hat{\cdot}$ and $\check{\cdot}$ give continuous isomorphism between $\mathcal{L}^s(^mX;Y)$ and $\mathcal{P}(^mX;Y)$ being one the inverse of the other. In particular for every homogeneous polynomial there is only one symmetric multilinear form associated to it.

Theorem 1.2.6 (Proposition 1.8 in [Din99]). *Given $T \in \mathcal{L}^s(^mX;Y)$ and $P = T \circ \Delta_m$, then $T = \check{P}$ and*

$$\|P\|_{\mathcal{P}(^mX;Y)} \leqslant \|T\|_{\mathcal{L}(^mX;Y)} \leqslant \frac{m^m}{m!}\|P\|_{\mathcal{P}(^mX;Y)}.$$

The following result compares the norm of a homogeneous polynomial with the norm of its associated symmetric multilinear form with some precise restriction on the elements.

Theorem 1.2.7 (Theorem 1 in [Har72]). *Given $P \in \mathcal{P}(^mX;Y)$ and $\check{P} \in \mathcal{L}(^mX;Y)$ the symmetric m-linear form associated to P, for any $k \in \{0,1,\dots,m\}$ it holds*

$$\sup_{u,v\in B_X} \|\check{P}(\underbrace{u,\dots,u}_{m-k},\overbrace{v,\dots,v}^{k})\|_Y \leqslant \frac{(m-k)!k!m^m}{(m-k)^{m-k}k^k m!}\|P\|_{\mathcal{P}(^mX;Y)}$$

1.3 Holomorphic functions

Now we center the attention on one of the main topics of this thesis, holomorphic functions over Banach spaces. We will define them, discuss some of their most important properties and their connection with the polynomials over Banach spaces.

Given X, Y Banach spaces over the complex field and $U \subset X$ an open set we say a function $f : U \to Y$ is Gâteaux-holomorphic if, given any triplet $\xi \in U, \eta \in X, \phi \in Y'$, the one complex variable mapping

$$\lambda \to \phi \circ f(\xi + \lambda\eta),$$

is defined and holomorphic in some neighborhood of 0 (in the traditional sense for a function of one complex variable). If $f : U \to Y$ is *Gâteaux-holomorphic* and continuous we say that mapping is holomorphic. As in the one dimentional case we will call a function holomorphic on the whole space X, *entire*. To denote the vector space of all the holomorphic functions over some open set $U \subset X$ with values in Y we use $H(U;Y)$ (or $H(U)$ if $Y = \mathbb{C}$). The set $H(U;Y)$ is in fact a $\mathbb{C}$-vector space.

The following theorem condenses three ways of understanding holomorphic functions over Banach spaces. For a proof of this fact and a deep insight on the theory of holomorphic function over Banach spaces we recommend [Din99] and [Muj10].

Theorem 1.3.1. *Given X, Y Banach spaces over the complex field, $U \subset X$ an open set and $f : U \to Y$ they are equivalent:*

- *f is holomorphic.*
- *f is Fréchet differentiable for every $x \in U$, i.e., it exists $df(x) \in \mathcal{L}(X,Y)$ such that*

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) - df(x)(h)}{\|h\|_X} = 0.$$

- *For every $x \in U$ there is $r = r(x) > 0$ such that in $B_r(x_0)$ the Taylor series of f converges uniformly, i.e., there are $\frac{d^m f(x_0)}{m!} \in \mathcal{P}(^m X, Y)$ for every $m \in \mathbb{N}$ such that*

$$f(x) = \sum_{m \geqslant 1} \frac{d^m f(x_0)}{m!}(x - x_0) + f(x_0),$$

for every $x \in B_r(x_0)$.

Given a holomorphic function $f : U \to Y$ and a point $x_0 \in U$, we will also use $P_m(f)(x_0) = \frac{d^m f(x_0)}{m!}$ (or simply $P_m(f)$ when the point x_0 is clearly determined) for the m-homogeneous part in the Taylor expansion.

From the third equivalence on Theorem 1.3.1 it becomes clear the deep connection between holomorphic functions and homogeneous polynomial in Banach spaces. We will exploit this relation all along this text. In this sense, a fundamental tool working with holomorphic function will be (as in the one dimensional case) the *Cauchy integral formula*. In this case, this formula states that, given a holomorphic function $f \in H(U;Y)$, and element $z \in U \subset X$ and other $x \in X$ it holds

$$\frac{d^m f(z)}{m!}(x) = \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{f(z + \lambda x)}{\lambda^{m+1}} d\lambda, \tag{1.10}$$

for every $m \in \mathbb{N}_0$, where $r > 0$ is a radius such that $z + \lambda x \in U$ for every $|\lambda| \leqslant r$.

There are many families of holomorphic functions considered in the literature, here we present those that we will study throughout the thesis. The first example is an already known family, given X, Y complex Banach spaces, for every $m \in \mathbb{N}$ we consider the family given by the m-homogeneos polynomials $\mathcal{P}(^m X; Y)$. The fact that it actually is a subset of the the of holomorphic functions from X to Y, i.e.,

$$\mathcal{P}(^m X; Y) \subset H(X; Y),$$

becomes clear from Theorem 1.3.1.

For any bounded set $U \subset X$ we denote by $H_\infty(U;Y)$ to the subset of $H(U;Y)$ given by the bounded holomorphic function from U to Y. The norm

$$\|f\|_{H_\infty(U;Y)} := \|f\|_U = \sup_{z \in U} \|f(z)\|_Y,$$

makes $\Big(H_\infty(U;Y), \|\cdot\|_{H_\infty(U;Y)}\Big)$ a Banach space. We will be especially interested in $H_\infty(B_X;Y)$, in particular for $Y = \mathbb{C}$, in this case we use the notation $H_\infty(B_X) :=$

$H_\infty(B_X;\mathbb{C})$. Next lemma states that, for any $m \in \mathbb{N}_0$, the projection

$$\begin{aligned} H_\infty(B_X;Y) &\to \mathcal{P}(^m X;Y) \\ f &\to \frac{d^m f(0)}{m!}, \end{aligned}$$

is a contraction.

Lemma 1.3.2. *For a couple of complex Banach spaces X, Y and an open balanced set $U \subset X$ set it holds*

$$\left\| \frac{d^m f(0)}{m!} \right\|_U \leqslant \|f\|_U,$$

for every $m \in \mathbb{N}_0$

The idea of the proof of Lemma 1.3.2 is essentially to use Cauchy integral formula (1.10).

Finally we will consider $H_b(X;Y) \subset H(H;Y)$ the set of entire functions of bounded type, i.e., the set of all entire functions over X with image on Y which are bounded over all bounded subsets of X. This set has a vector spaces structure. Even more, for $f \in H_B(X;Y)$ and any positive integer n, $q_n(f) := \|f\|_{nB_X}$ defines a seminorm. The space $H_b(X;Y)$ is a Frchet space considering the family of seminorms $\{q_n : n \in \mathbb{N}\}$. A mapping f is an entire functions of bounded type if and only if

$$\lim_{m\to\infty} \left\| \frac{d^m f(0)}{m!} \right\|_{B_X}^{1/m} = 0, \tag{1.11}$$

(see [Muj10, Corollary 7.4]) and, for every $r > 0$, $p_r(f) := \sum_{m\geqslant 0} r^m \left\| \frac{d^m f(0)}{m!} \right\|_{B_X}$ is a seminorm in $H_b(X;Y)$. The family of seminorms $\{p_r : r > 0\}$ gives in $H_b(X;Y)$ the same Fréchet space structure. When we have $Y = \mathbb{C}$ we will just write $H_b(X)$ to refer to this space.

In the finite dimensional case, a complex function is holomorphic on an open set if and only if it is analytic there.

Theorem 1.3.3. *Given an open set $U \subset \mathbb{C}^n$ any mapping $f : U \to \mathbb{C}$ is holomorphic on U if and only if is analitic on U. In this case, given $z \in U$, there are $r_1, \ldots, r_n > 0$ (depending on z) such that*

$$f(w) = \sum_{\alpha\in\mathbb{N}_0^n} a_\alpha(f)(z)(w - z)^\alpha.$$

for every $w \in z + (r_1, \ldots, r_n) \cdot \mathbb{D}^n$. Even more, its coefficients may be calculated by the formula

$$a_\alpha(f)(z) = \frac{1}{(2\pi i)^n} \int_{|\xi_1 - z_1| = \rho_1} \cdots \int_{|\xi_n - z_n| = \rho_n} \frac{f(\xi_1, \ldots, \xi_n)}{(\xi_1 - z_1)^{\alpha_1+1} \cdots (\xi_n - z_n)^{\alpha_n+1}} d\xi. \tag{1.12}$$

1.4 Complex interpolation

The complex interpolation method developed by Calderón is a very useful tool in the study of bounded linear operators between Banach spaces and, in general, inequalities in these spaces. This method allows to get information about the linear operators between a family of spaces by knowing how these operators behave in the extreme spaces of the family.

We will say that two Banach spaces X, Y form an interpolation couple if there is a Hausdorff topological vector space Λ and injective and continous operators

$$i_X : X \hookrightarrow \Lambda, \text{ y } i_Y : Y \hookrightarrow \Lambda.$$

Whenever (X, Y) is an *interpolation couple* we consider the Banach space given by their sum

$$X + Y := \Big\{ i_X(x) + i_Y(y) : x \in X, y \in Y \Big\}$$

with the norm

$$\|u\|_{X+Y} := \inf\{\|x\|_X + \|y\|_Y : x \in X, y \in Y, u = i_X(x) + i_Y(y)\};$$

and the Banach spaces given by their intersection

$$X \cap Y := i_X(X) \cap i_Y(Y),$$

with the norm $\|u\|_{X\cap Y} := \max \Big\{ \|i_X^{-1}(u)\|_X, \|i_Y^{-1}(u)\|_Y \Big\}$.

The inclusion

$$\begin{aligned} \iota : X \cap Y &\hookrightarrow X + Y, \\ u &\mapsto u, \end{aligned}$$

is an isometry, this allows to define an *intermediate space* E for the interpolation couple (X, Y) as any Banach space such that $X \cap Y \subset E \subset X + Y$ as Banach spaces (i.e., $\|u\|_{X+Y} \leqslant \|u\|_E \leqslant \|u\|_{X\cap Y}$). An interpolation space between X and Y is any intermediate space E such that, given $T \in \mathcal{L}(X + Y)$ fulfilling $T|_X \in \mathcal{L}(X)$ and $T|_Y \in \mathcal{L}(Y)$, it holds that $T|_E \in \mathcal{L}(E)$.

To define complex interpolation we will use the complex strip $B := \{z = a + bi : 0 < a < 1\}$. Let (X, Y) be an interpolation couple, we say a function $f : \overline{B} \to X + Y$ fulfills the $(*)$ conditions if

1. f is continuous on $\overline{B}$;

2. f is holomorphic on B;

3. $t \mapsto f(it)$ is continuous and bounded from $\mathbb{R}$ to X, $t \mapsto f(1 + it)$ continuous and bounded from $\mathbb{R}$ to Y.

Consider the Banach space given by the complex functions attaining the $(*)$ conditions, i.e.,

$$\mathcal{F}(X,Y) := \Big\{ f : \overline{B} \to X+Y : \text{ fulfilling the } (*) \text{ conditions} \Big\},$$

with the norm given by $\|f\|_{\mathcal{F}(X,Y)} := \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_X, \sup_{t\in\mathbb{R}} \|f(1+it)\|_X\}$.

Given $0<\theta<1$ consider

$$\mathcal{N}_\theta(X,Y) = \Big\{ f \in \mathcal{F}(X,Y) : f(\theta) = 0 \Big\},$$

which is a closed subspace of $\mathcal{F}(X,Y)$. The intermediate space given by the complex interpolation method in θ is given by

$$[X,Y]_\theta := \frac{\mathcal{F}(X,Y)}{\mathcal{N}_\theta(X,Y)},$$

endowed with the quotient norm.

Given $1 \leqslant p_0, p_1 \leqslant \infty$ a classic result of Riesz and Thorin implies that for every $0<\theta<1$ it holds

$$[Lp_0(U,\mu), L_{p_1}(U,\mu)]_\theta = L_p(\mu), \tag{1.13}$$

isometrically, where $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ for every (U,μ) measure space.

Theorem 1.4.1 (Multilinear interpolation Theorem)**.** *For $m \in \mathbb{N}$ let $(X_0^1, X_1^1), \ldots, (X_0^m, X_1^m), (Y_0, Y_1)$ be interpolation couples and $T \in \mathcal{L}(X_0^1, \ldots, X_0^m; Y_0) \cap \mathcal{L}(X_1^1, \ldots, X_1^m; Y_1)$. Then for every $\theta \in (0,1)$ it holds $T \in \mathcal{L}([X_0^1, X_1^1]_\theta, \ldots, [X_0^m, X_1^m]_\theta; [Y_0, Y_1]_\theta)$ with*

$$\|T\|_{\mathcal{L}([X_0^1,X_1^1]_\theta,\ldots,[X_0^m,X_1^m]_\theta;[Y_0,Y_1]_\theta)} \leqslant \|T\|^{1-\theta}_{\mathcal{L}(X_0^1,\ldots,X_0^m;Y_0)} \|T\|^{\theta}_{\mathcal{L}(X_1^1,\ldots,X_1^m;Y_1)}.$$

1.5 Dirichlet series

To understand the thesis results we do not need this section. Nevertheless it is necessary to understand in a deeper way its motivation. It is included to give the definitions and background involved in some historical facts mentioned in Chapter 2.

Given sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ we may define formally the induced *Dirichlet series*

$$D := \sum_{n\geqslant 1} a_n \frac{1}{n^s}.$$

We will denote $\mathcal{D}$ to the set of all Dirichlet series. Again formally we may define linear structure on $\mathcal{D}$, namely

$$\sum_{n\geqslant 1} a_n \frac{1}{n^s} + \sum_{n\geqslant 1} b_n \frac{1}{n^s} := \sum_{n\geqslant 1} (a_n + b_n) \frac{1}{n^s},$$
$$\lambda \left(\sum_{n\geqslant 1} a_n \frac{1}{n^s} \right) := \sum_{n\geqslant 1} (\lambda a_n) \frac{1}{n^s},$$

and an algebraic structure given by product

$$\left(\sum_{n\geqslant 1} a_n \frac{1}{n^s}\right)\left(\sum_{n\geqslant 1} b_n \frac{1}{n^s}\right) := \sum_{n\geqslant 1}\left(\sum_{km=n} a_k b_m\right)\frac{1}{n^s}$$

which is usually called "Dirichlet" multiplication. These series play a key role in number theory. The most prominent example of a Dirichlet series is the well known Riemann ζ-function

$$\zeta = \sum_{n\geqslant 1} \frac{1}{n^s}.$$

For $\sigma \in \mathbb{R}$ it will be useful the notation $[Re > \sigma] := \{z \in \mathbb{C} : Re(z) > \sigma\}$, $[Re < \sigma]$ and $[Re = \sigma]$ are defined analogously. Given a Dirichlet series D and $s \in \mathbb{C}$ such that the complex series $D(s) = \sum_{n\geqslant 1} a_n \frac{1}{n^s}$ converges it is well known that $D(s')$ also converges for every $s' \in [Re > Re(s)]$ (see [DGMSP19, Theorem 1.1]). We may then consider, given $D \in \mathcal{D}$, its *Abscissa of convergence* to be $[Re = \sigma_c(D)]$ where

$$\sigma_c(D) := \inf\{\sigma \in \mathbb{R} : D \text{ converges in } [Re > \sigma]\}.$$

The following classic result of Dirichlet series theory is the first step in the study of this objects from an analytical point of view.

Theorem 1.5.1. *Let D be a Dirichlet series not everywhere divergent. Then it converges in $[Re > \sigma_c(D)]$ and diverges on $[Re < \sigma_c(D)]$. Even more, the mapping*

$$\begin{aligned} D : [Re > \sigma_c] &\to \mathbb{C} \\ s &\mapsto D(s) = \sum_{n\geqslant 1} a_n \frac{1}{n^s}, \end{aligned}$$

is holomorphic.

Notice that, for $D = \sum_{n\geqslant 1} a_n \frac{1}{n^s}$, its region of absolute convergence is exactly the region of convergence for $\sum_{n\geqslant 1} |a_n| \frac{1}{n^s}$. This gives that the region of absolutely convergence for a given Dirichlet series is again half plane. We may consider the *abscissa of absolute convergence* for $D \in \mathcal{D}$ to be $[Re = \sigma_a(D)]$ where

$$\sigma_a(D) := \inf\{\sigma \in \mathbb{R} : D \text{ converges absolutely in } [Re > \sigma]\}.$$

It is plain that $\sigma_c(D) \leqslant \sigma_a(D)$ for every $D \in \mathcal{D}$.

A third abscissa plays an important role in this part of the theory of Dirichlet series. Notice that given $D \in \mathcal{D}$ we may look upon the sequence of functions defined by its partial sums

$$(D_N)_{N\in\mathbb{N}} := \left(\sum_{n=1}^{N} a_n \frac{1}{n^s}\right)_{N\in\mathbb{N}}$$

and study the region of uniform convergence of this sequence. In this way it arises the *abscissa of uniform convergence* for some $D \in \mathcal{D}$ as $[Re = \sigma_u(D)]$ where

$$\sigma_u(D) := \inf\left\{\sigma \in \mathbb{R} : (D_N)_{N\in\mathbb{N}} \text{ converges absolutely to } D \text{ in } [Re > \sigma]\right\}.$$

It is not hard to see that $\sigma_c(D) \leqslant \sigma_u(D) \leqslant \sigma_a(D)$ for every $D \in \mathcal{D}$.

Notice that the half-planes of convergence, uniform convergence and absolute convergence are the Dirichlet series version of the disc of convergence, uniform convergence and absolute convergence for Taylor series. It is a classic result that in the Taylor series case all of those disks are the same. Some natural questions arise trying to understand this approach to the Dirichlet series,

- Will the half planes of different kind of convergence be always the same?
- If not, how big is the distance between the abscissas?

For a certain Dirichet series all of them may coincide but in general, they don't. Take for example $\tilde{D} = \sum_{n\geqslant 1}(-1)^n n^{-s}$. This series converges for $[Re > 0]$ but $\sigma_a(\tilde{D}) > 1$ so $\sigma_a(\tilde{D}) - \sigma_c(\tilde{D}) \geqslant 1$. On the other hand, given $D = \sum_{n\geqslant 1} a_n n^{-s}$ with $\sigma_c(D) < \infty$, for $s_0 = \sigma_0 + it \in [Re > \sigma_c(D)]$ as $D(s_0)$ converges $(|a_n|n^{-\sigma_0})_{n\geqslant 1}$ is bounded by some $K > 0$. Given any $\varepsilon > 0$ we have

$$\sum_{n\geqslant 1}\left|a_n \frac{1}{n^{s_0+1+\varepsilon}}\right| = \sum_{n\geqslant 1} |a_n| \frac{1}{n^{\sigma_0+1+\varepsilon}} \leqslant K \sum_{n\geqslant 1} \frac{1}{n^{1+\varepsilon}} < \infty.$$

Then we have $\sigma_a(D) \leqslant \sigma_c(D) + 1$. This proves the following theorem.

Theorem 1.5.2. $\sup_{D\in\mathcal{D}} \sigma_a(D) - \sigma_c(D) = 1.$

As we have seen the convergence regions for Dirichlet series have a new behaviour with respect to Taylor series in this sense, and it is worth studying it.

In the early years of the 20th century Harald Bohr began a study of the Dirichlet series. In particular he was very interested in determining the size of the gap between the regions of convergence, uniform convergence and absolute convergence. With this general problem in mind he developed a number of tools that allowed him to prove that

$$\sup_{D\in\mathcal{D}} \sigma_u(D) - \sigma_c(D) = 1.$$

The last of the gaps,

$$\mathcal{S} := \sup_{D\in\mathcal{D}} \sigma_a(D) - \sigma_u(D),$$

historically required much more effort to determine. The first step to understand this problem was to give the Dirichlet series a Banach space structure. Consider

$$\mathcal{H}_\infty := \{D \in \mathcal{D} : \sigma_c(D) \leqslant 0\} \subset H([Re > 0]),$$

this results a Banach algebra endowed with the norm given by

$$\|D\|_{\mathcal{H}_\infty} := \sup_{s\in[Re>0]} |D(s)|.$$

Proposition 1.5.3 (Proposition 1.24 in [DGMSP19]). $S = \sup_{D \in \mathcal{H}_\infty} \sigma_a(D)$.

One of the main ideas involved in solving this problem consists in the use of the so-called *Bohr transform*: a way that Bohr found to build a bridge between the Dirichlet series and the holomorphic functions of many infinite variables. This transform is defined in the following way

$$\begin{aligned} \mathfrak{B} : H_\infty(B_{c_0}) &\to \mathcal{H}_\infty \\ f &\mapsto \sum_{n=1}^{\infty} a_n n^{-s}, \end{aligned} \tag{1.14}$$

where, given $n = p_1^{\alpha_1} \cdots p_N^{\alpha_N}$ written in terms of its prime number decomposition and $(p_i)_{i \in \mathbb{N}}$ is the sequence of ordered prime numbers,

$$a_n := \frac{1}{(2\pi i)^N} \int_{|z_1|=r} \cdots \int_{|z_N|=r} \frac{f(z_1, \ldots, z_n, 0, \ldots)}{z_1^{\alpha_1+1} \cdots z_N^{\alpha_N+1}} dz_1 \cdots dz_N.$$

In other words $a_n = c_\alpha(f)$ where $n = p^\alpha := p_1^{\alpha_1} \cdots p_N^{\alpha_N}$.

Its inverse is called the Bohr lift

$$\begin{aligned} \mathfrak{B}^{-1} : \mathcal{H}_\infty &\to H_\infty(B_{c_0}) \\ \sum_{n=1}^{\infty} a_n n^{-s} &\mapsto \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_{p^\alpha} z^\alpha, \end{aligned}$$

where again $p^\alpha := p_1^{\alpha_1} \cdots p_N^{\alpha_N}$ for every $\alpha = (\alpha_1, \ldots, \alpha_N, 0, \ldots) \in \mathbb{N}_0^{(\mathbb{N})}$.

Theorem 1.5.4. *The Bohr transform is an isometric ismorphism between the Banach algebras $\mathcal{H}_\infty$ and $H_\infty(B_{c_0})$.*

As the m-homogeneous polynomial are crucial to understand the holomorphic functions, this point of view suggests the definition of the m-homogeneous Dirichlet polynomial. Recall that, for every integer n, $\Omega(n)$ is the numbers of prime divisors counted with multiplicity. The set of m-homogeneous Dirichlet series is defined by

$$\mathcal{D}_m := \left\{ \sum_n a_n n^{-s} \in \mathcal{D} : a_n \neq 0 \implies \Omega(n) = m. \right\}$$

An intermediate problem in the way to determine $\mathcal{S}$ is to understand its m-homogenous version, the gap

$$\mathcal{S}_m := \sup_{D \in \mathcal{D}_m} \sigma_a(D) - \sigma_u(D)$$

To give Banach structure to $\mathcal{D}_m$ we define the subspace of m-homogeneous Dirichlet series which converge on the half-plane $[Re > 0]$ as

$$\mathcal{H}_\infty^m := \mathcal{D}_m \cap \mathcal{H}_\infty = \left\{ \sum_n a_n n^{-s} \in \mathcal{H}_\infty : a_n \neq 0 \implies \Omega(n) = m, \right\}$$

endowed with the norm given in $\mathcal{H}_\infty$.

Theorem 1.5.5. *The Bohr transform is an isometric Banach space isomorphism between* $\mathcal{H}_\infty^m$ *and* $\mathcal{P}(^m\ell_\infty)$.

A very useful tool to establish the value of $\mathcal{S}_m$ will be the following m-homogeneous version of Proposition 1.5.3.

Proposition 1.5.6. $\mathcal{S}_m = \sup_{D \in \mathcal{H}_\infty^m} \sigma_a(D)$.

Chapter 2

Coefficients summability

In 1930 Littlewood [Lit30] proved his celebrated (and nowadays classical) 4/3-inequality. It states that, given a bilinear form $B : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$, the following holds

$$\left(\sum_{i,j=1}^{n} |B(e_i, e_j)|^{4/3} \right)^{3/4} \leqslant \sqrt{2} \|B\|_{\mathcal{L}(^2\ell_\infty^n)}.$$

Even more, he proved that the exponent 4/3 cannot be improved, i.e., for every $r < 4/3$ it is impossible to have an analogous inequality with a constant independent of the number of variables. This was the first of many "coefficient summing inequalities" of this kind. Those inequalities proved to be very useful to solve problems in a wide variety of branches of mathematics.

In the early years of the 20^{th} century Harald Bohr [Boh13] began a study of the theory of Dirichlet series. In particular he was very interested in determining the size of the gap between the regions where those mappings converge in different ways. One of the hardest and main problems he presented in this field was to determine the biggest possible gap between the abscissa of uniform convergence $\sigma_u(D)$ and absolute convergence $\sigma_a(D)$ for Dirichlet series,

$$\mathcal{S} = \sup_{D \in \mathcal{D}} \sigma_a(D) - \sigma_u(D),$$

where $\mathcal{D}$ stand for the set of Dirichlet series. Bohr managed to show that $\mathcal{S} \leqslant \frac{1}{2}$, among other many contributions to this theory, but he could not give the precise value of $\mathcal{S}$.

He was able to translate this problem to the language of holomorphic functions over infinitely many variables by the so called *Bohr transform*. In this context, an intermediate problem, is to determine the size of the gap for m-homogeneous Dirichlet series

$$\mathcal{S}_m = \sup_{D \in \mathcal{D}_m} \sigma_a(D) - \sigma_u(D).$$

With this problem in mind Bohnenblust and Hille reached a novel generalization of the 4/3-inequality [BH31] that allowed them to give the final value for the gap.

2.1 Some summability results

Bohnenblust and Hille were interested in finding the value of the gap between the abscissa of uniform and absolute convergence for Dirichlet series. To do so they showed a sufficiently good way to control the sum of the coefficients of any m-linear form to the power $\frac{2m}{m+1}$ by its uniform norm in ℓ_∞.

Theorem 2.1.1 (Multilinear Bonhenblust-Hille inequality)**.** *Given $m, n \in \mathbb{N}$, for every m-linear form $T : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \mathbb{C}$ there is $C_m > 0$ depending only on m (not on n) such that*

$$\left(\sum_{1 \leqslant i_1, \dots, i_m \leqslant n} |T(e_{i_1}, \dots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leqslant C_m \|T\|_{\mathcal{L}(^m\ell_\infty^n)}. \tag{2.1}$$

Moreover the exponent $\frac{2m}{m+1}$ is optimal.

In general, fixed $m, n \in \mathbb{N}$ and $1 \leqslant r < \infty$, we may consider the constant $C_{r,m}(n) > 0$ such that for every m-linear form $T : \mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \mathbb{C}$ it holds

$$\left(\sum_{1 \leqslant i_1, \dots, i_m \leqslant n} |T(e_{i_1}, \dots, e_{i_m})|^{r} \right)^{1/r} \leqslant C_{m,r}(n) \|T\|_{\mathcal{L}(^m\ell_\infty^n)}.$$

This constant $C_{m,r}(n)$ may depend on m and r but also on the number of complex variables n. In Theorem 2.1.1 optimal means that if $C_{m,r}(n) = C_{m,r}$ does not depend on n then $r \geqslant \frac{2m}{m+1}$. In other words, for $r < \frac{2m}{m+1}$ the dependence on n of $C_{m,r}$ becomes explicit, in particular it holds $C_{m,r}(n) \to \infty$ when the number of variables n goes to infinity.

We denote by BH_m^{mult} the best constant C_m in Theorem 2.1.1. The original proof due to Bohnenblust and Hille gives the bound $B_m^{mult} \leqslant m^{\frac{m+1}{2m}} 2^{\frac{m-1}{2}}$.

In order to reach the value of the gap between uniform and absolute convergence in Dirichlet series, Bohnenblust and Hille needed a polynomial version of this inequality. To obtain that inequality they developed of polarization and from Theorem 2.1.1 achieved the following result.

Theorem 2.1.2 (Polynomial Bonhenblust-Hille inequality)**.** *Given $m, n \in \mathbb{N}$, for every homogeneous polynomial $P \in \mathcal{P}(^m\mathbb{C}^n)$ with coefficients $(a_\alpha(P))_{\alpha \in \mathbb{N}_0^n}$ there is a constant $C_m >$ 0 depending only on m such that*

$$\left(\sum_{\alpha \in \Lambda(m,n)} |a_\alpha(P)|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leqslant C_m \|P\|_{\mathcal{P}(^m\ell_\infty^n)}. \tag{2.2}$$

Moreover, $\frac{2m}{m+1}$ is optimal.

Proof. Let $\check{P}$ be the symmetric m-linear form associated to P. By Theorem 2.1.1, Remark 1.2.5 and using the polarization formula in Theorem 1.2.6 it holds

$$\begin{aligned}\left(\sum_{\alpha\in\Lambda(m,n)}|a_\alpha(P)|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}} &= \left(\sum_{\mathbf{j}\in\mathcal{J}(m,n)}|\check{P}(e_{j_1},\ldots,e_{j_m})||[\mathbf{j}]||^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}}\\ &= \left(\sum_{\mathbf{j}\in\mathcal{J}(m,n)}\frac{|[\mathbf{j}]|^{\frac{2m}{m+1}}}{|[\mathbf{j}]|}\sum_{\mathbf{i}\in[\mathbf{j}]}|\check{P}(e_{i_1},\ldots,e_{i_m})|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}}\\ &\leqslant m!^{\frac{m-1}{m+1}}\left(\sum_{1\leqslant i_1\leqslant\ldots\leqslant i_m\leqslant n}|\check{P}(e_{i_1},\ldots,e_{i_m})|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}}\\ &\leqslant m^m BH_m^{mult}\|\check{P}\|_{\mathcal{L}(^m\ell_\infty^n)} \leqslant m^{m+\frac{m+1}{2m}}2^{\frac{m-1}{2}}\|P\|_{\mathcal{P}(^m\ell_\infty^n)},\end{aligned}$$

where we used $|[\mathbf{j}]| \leqslant m! \leqslant m^m$ for every $\mathbf{j}\in\mathcal{J}(m,n)$ and the previously mentioned bound for BH_m^{mult}. We will later give examples of polynomials that prove the optimality of the exponent $\frac{2m}{m+1}$. □

As in the multilinear version, in Theorem 2.1.2 optimal means that, if there exists some $C_{r,m}(n) > 0$ such that for every $P \in \mathcal{P}(^m\mathbb{C}^n)$ it holds

$$\left(\sum_{\alpha\in\Lambda(m,n)}|a_\alpha(P)|^r\right)^{1/r} \leqslant C_{m,r}(n)\|P\|_{\mathcal{P}(^m\ell_\infty^n)},$$

and $C_{m,r}(n) = C_{m,r}$ does not depend on n then $r \geqslant \frac{2m}{m+1}$. As before, for $r < \frac{2m}{m+1}$ the dependence on the number of variables becomes explicit and

$$C_{m,r}(n) \to \infty \text{ as } n \to \infty,$$

as we will show later in this chapter. We denote by BH_m^{pol} to the best constant C_m in Theorem 2.1.2. Many efforts have been made to find good bounds for this constant. From Theorem 2.1.2 we have $BH_m^{pol} \leqslant m^{m+\frac{m+1}{2m}}2^{\frac{m-1}{2}}$. This is not even close to the best known bound for this constant, one of the most important contribution to the study of it is done in [DFOC+11] where the authors prove that

$$BH_m^{pol} \leqslant \left(1+\frac{1}{m-1}\right)^{m-1}\sqrt{m}2^{\frac{m-1}{2}}.$$

Thanks to Theorem 2.1.2, the fact that the Bohr transform is an isometry between $\mathcal{H}_\infty^m$ and $\mathcal{P}(^m\ell_\infty)$ and the characterization of $\mathcal{S}_m = \sup_{D\in\mathcal{H}_\infty^m}\sigma_a(D)$, Bonhenblust and Hille were able to show that

$$\mathcal{S}_m = \frac{m-1}{2m},$$

and then as $\mathcal{S}_m \leqslant \mathcal{S} \leqslant \frac{1}{2}$ for every $m \in \mathbb{N}$, they concluded

$$\mathcal{S} = \frac{1}{2}.$$

The Bohnenblust-Hille inequality in Theorem 2.1.2 explores the best number $1 \leqslant r \leqslant \infty$ such that the space of m-homogeneous polynomials bounded on the ball of ℓ_∞ may support the ℓ_r-norm of the coefficients. Notice that, given $P \in \mathcal{P}(^m\ell_\infty)$, its coefficients sequence $(a_\alpha(P))_{\alpha\in\Lambda(m)}$ may have infinitely nonzero elements and, given $1 \leqslant r \leqslant \infty$ it may or may not be in the space

$$\ell_r(\Lambda(m)) := \left\{ (a_\alpha)_{\alpha\in\Lambda(m)} \subset \mathbb{C} : \left(\sum_{\alpha\in\Lambda(m)} |a_\alpha|^r \right)^{1/r} < \infty \right\}.$$

A qualitative way of reading Theorem 2.1.2 is the following corollary

Corollary 2.1.3. *Let $m \in \mathbb{N}$, for every m-homogeneous polynomial $P \in \mathcal{P}(^m\ell_\infty)$ it holds that $(a_\alpha(P))_{\alpha\in\Lambda(m)} \in \ell_{\frac{2m}{m-1}}(\Lambda(m))$. Even more, for every $1 \leqslant r < \frac{2m}{m-1}$ there is some $P \in \mathcal{P}(^m\ell_\infty)$ such that $(a_\alpha(P))_{\alpha\in\Lambda(m)} \notin \ell_r(\Lambda(m))$.*

Now we will simply prove the first statement of Corollary 2.1.3 using Theorem 2.1.2. For the second statement we will need to show the existence of some polynomial in $\mathcal{P}(^m\ell_\infty)$ with unbounded ℓ_r norm for its coefficients. In the original proof of their result, Bohnenblust and Hille constructed a family of polynomials that fulfill that condition. We will follow another path, which uses probability theory to assure the existence of some kind of extreme polynomials. Those will allow us to get a better understanding of a more general problem and we will present them latter in Subsection 2.2.1.

Proof of the first statement in Corollary 2.1.3. For $P \in \mathcal{P}(^m\ell_\infty)$ consider its composition with the projection to the first n coordinates $P_n = P \circ \pi_n \in \mathcal{P}(^m\mathbb{C}^n)$. By Theorem 2.1.2, using that $\|P_n\|_{\mathcal{P}(^m\ell_\infty^n)} \leqslant \|P\|_{\mathcal{P}(^m\ell_\infty)}$ (by Corollary 1.2.3) and Remark 1.2.4 we have

$$\left(\sum_{\alpha\in\Lambda(m,n)} |a_\alpha(P)|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leqslant C_m \|P_n\|_{\mathcal{P}(^m\ell_\infty^n)} \leqslant C_m \|P\|_{\mathcal{P}(^m\ell_\infty)},$$

where $C_m > 0$ and does not depend on n. Taking the limit on n going to ∞ we have what we needed. □

Now it seems natural to ask if we can change the space ℓ_∞ in Corollary 2.1.3 with other ℓ_p space and get a similar result. This is the first of a number of questions we will try to answer in this chapter. As the trigger for the Bonhenblust-Hille inequality was the Littelwood 4/3-inequality, the first attempt to answer the previous question was a generalization of Littelwood's result given by the legendary couple he made with Hardy

[HL34]. This generalization considered bilinear forms $B : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ and studied inequalities of the form

$$\left(\sum_{i,j=1}^{n} |B(e_i, e_j)|^q\right)^{1/q} \leqslant K\|B\|_{\mathcal{L}(\ell_{p_1}^n, \ell_{p_2}^n)}.$$

They were particularly interested in the conditions on $1 \leqslant q, p_1, p_2 \leqslant \infty$ to guarantee the existence of $K > 0$ independent of n. A half of century latter the bilinear inequalities of Hardy and Littlelwood inspired Praciano-Pereira [PP81] to attack the multilinear version of that problem. This problem was also studied by Dimant and Sevilla-Peris in [DSP16].

Given an m-homogeneous polynomial $P(z) = \sum_{\alpha \in \Lambda(m,n)} a_\alpha z^\alpha$ in n variables we denote the ℓ_r-norm of its coefficients by

$$|P|_q := \Big(\sum_{\alpha \in \Lambda(m,n)} |a_\alpha|^q\Big)^{1/q}.$$

Another norm, related to the coefficients is the so-called Bombieri q-norm defined in [BBEM90]:

$$[P]_q := \Big(\sum_{\alpha \in \Lambda(m,n)} \big(\frac{\alpha!}{m!}\big)^{q-1} |a_\alpha|^q\Big)^{1/q}.$$

The relation between the these coefficients-norms is given by the following inequalities (see [BBEM90]):

$$(m!)^{\frac{1}{q}-1} |P|_q \leqslant [P]_q \leqslant |P|_q. \tag{2.3}$$

For $1 \leqslant p, q \leqslant \infty$ we will say it holds a *polynomial Hardy-Littlewood type inequality* whenever, given $m, n \in \mathbb{N}$ there is some $C_{m,p,q} > 0$ independent of n such that

$$|P|_q \leqslant C_{m,p,q} \|P\|_{\mathcal{P}(^m \ell_p^n)}, \tag{2.4}$$

for every $P \in \mathcal{P}(^m\mathbb{C}^n)$. Whenever a Hardy-Littlewood inequality holds for $1 \leqslant p, q \leqslant \infty$ we say that (p, q) forms a *Hardy-Littlewood pair.*

The following lemma, which is an immediate consequence of the Cauchy integral formula, will be very important in the development of Theorem 2.1.7 which is a key tool for this thesis. It will also be useful to notice (p, ∞) forms a *Hardy-Littlewood pair* for every $1 \leqslant p \leqslant \infty$. For some $1 \leqslant p \leqslant \infty$ this will be the only possible Hardy-Littlewood pair.

Lemma 2.1.4. *Given $1 \leqslant p \leqslant \infty$, $m, n \in \mathbb{N}$ and $\alpha \in \Lambda(m, n)$, for every $P \in \mathcal{P}(^m\mathbb{C}^n)$ it holds*

$$|a_\alpha(P)| \leqslant \left(\frac{m^m}{\alpha^\alpha}\right)^{1/p} \|P\|_{\mathcal{P}(^m \ell_p^n)}. \tag{2.5}$$

Proof. Given $u = \frac{1}{m^{1/p}} \alpha^{1/p} \in B_{\ell_p^n}$ by Cauchy integral formula in equation (1.10) we have

$$\begin{aligned} |a_\alpha(P)| &\leqslant \frac{1}{(2\pi i)^n} \int_{|z_1|=u_1} \cdots \int_{|z_n|=u_n} \frac{|P(z)|}{|z_1^{\alpha_1+1} \cdots z_n^{\alpha_n+1}|} dz \\ &\leqslant \frac{1}{u_1^{\alpha_1} \cdots u_n^{\alpha_n}} \|P\|_{\mathcal{P}(^m \ell_p^n)} \leqslant \left(\frac{m^m}{\alpha^\alpha}\right)^{1/p} \|P\|_{\mathcal{P}(^m \ell_p^n)}, \end{aligned}$$

as $z \in B_{\ell_p^n}$ for every $|z| = u$. $\square$

In some problems it will be necessary to understand, given a multi-index $\alpha \in \Lambda(M, n)$, how to bound $\frac{M^M}{\alpha^\alpha}$. For example, if $n \geqslant M$ we can pick

$$\alpha = (\underbrace{1, \ldots, 1}_{M}, 0, \ldots, 0),$$

in this case $\frac{M^M}{\alpha^\alpha} = M^M$. On the other hand if $\alpha = (M, 0, \ldots, 0)$ a simple calculation tells us that $\frac{M^M}{\alpha^\alpha} = 1$. In general $\alpha^\alpha \geqslant 1$ for every $\alpha \in \Lambda(M, n)$, then

$$\sup_{\alpha \in \Lambda(M,n)} \frac{M^M}{\alpha^\alpha} \leqslant M^M, \tag{2.6}$$

and the equality holds when $n \geqslant M$. Now by Lemma 2.1.4 and equation (2.6) we have for every $P \in \mathcal{P}(^m\mathbb{C}^n)$

$$|P|_\infty \leqslant m^m \|P\|_{\mathcal{P}(\ell_p^n)}. \tag{2.7}$$

Remark 2.1.5. For $1 \leqslant p \leqslant m$ there is no $q < \infty$ such that (p, q) is a Hardy-Littlelwood pair.
Indeed, taking $P = \sum_{j=1}^n z_j^m$, for every $z \in \mathbb{C}^n$ as $m \leqslant p$ it holds

$$|P(z)| \leqslant \sum_{j=1}^n |z_j|^m = \|z\|_{\ell_m}^m \leqslant \|z\|_{\ell_p}^m,$$

then we have

$$|P|_q = n^{\frac{1}{q}} \text{ and } \|P\|_{\mathcal{P}(^m\ell_p^n)} \leqslant 1.$$

This proves that there is no constant independent of n as in (2.4).

The following result gathers [DSP16, Proposition 4.1] and [PP81, Theorem A and Theorem B] completing the description of the Hardy-Littlelwood type inequalities.

Theorem 2.1.6 (Polynomial Hardy-Littlewood type inequalities)**.** *Fixed $m, n \in \mathbb{N}$ and $m < p \leqslant \infty$, there is a constant $C_{m,p} > 0$ depending only on m and p (not on n) such that for every m-homogeneous polynomial in n-complex variables P we have:*

$$(i) \qquad |P|_{\frac{p}{p-m}} \leqslant C_{m,p} \, \|P\|_{\mathcal{P}(^m\ell_p^n)} \qquad \text{for } m \leqslant p \leqslant 2m,$$

$$(ii) \qquad |P|_{\frac{2mp}{mp+p-2m}} \leqslant C_{m,p} \, \|P\|_{\mathcal{P}(^m\ell_p^n)} \qquad \text{for } 2m \leqslant p.$$

Again the exponents $\frac{p}{p-m}$ and $\frac{2mp}{mp+p-2m}$ in the above inequalities are the best possible. Observe that, in the limit case ($p = \infty$) we recover the classical Bohnenblust-Hille exponent $\frac{2m}{m+1}$.

By Remark 2.1.5 and Theorem 2.1.6 we have a full description of the condition of existence of Hardy-Littlelwood inequalities. Nevertheless we may get more useful inequalities modifying the coefficients norm on the leftmost side of the inequalities or accepting a wider variety of spaces to take the uniform norm on the right side. One example of this kind of modification on the left side is next theorem due to Bayart, Defant and Schlüters [BDS19]. Here we present a slight modification of their result. This will allow us to have better bounds in future applications.

Theorem 2.1.7 (Bayart-Defant-Shlüters inequality)**.** *Let* $1 \leqslant p \leqslant \infty$*,* $m, n \in \mathbb{N}$ *and* $P \in \mathcal{P}(^m\mathbb{C}^n)$*. Then for each* $\mathbf{j} \in \mathcal{J}(m-1,n)$ *with associated multi-index* $\alpha(\mathbf{j}) \in \Lambda(m-1,n)$ *we have*

$$\left(\sum_{k=j_{m-1}}^{n} |c_{(\mathbf{j},k)}(P)|^{p'}\right)^{\frac{1}{p'}} \leqslant em\left(\frac{(m-1)^{m-1}}{\alpha(\mathbf{j})^{\alpha(\mathbf{j})}}\right)^{\frac{1}{p}} \|P\|_{\mathcal{P}(^m\ell_p^n)}. \tag{2.8}$$

Theorem 2.1.7 will be a fundamental piece in the developments of Chapters 3, 5, 6 and 8. As it will be oftelny used, we will name the inequality in Theorem 2.1.7 *Bayart-Defant-Schlüters inequality* or sometimes *DBS inequality* for short.

We will follow [BDS19] and give the proof of Theorem 2.1.7 for completeness.

For $\mathbf{j} \in \mathcal{J}(m-1,n)$ and $\alpha(\mathbf{j}) \in \Lambda(m-1,n)$ its associated multi-exponent it holds $\frac{(m-1)^{m-1}}{\alpha(\mathbf{j})^{\alpha(\mathbf{j})}} \leqslant e^{m-1}\frac{(m-1)!}{\alpha(\mathbf{j})!} = e^{m-1}|[\mathbf{j}]|$, then by Theorem 2.1.7 we have

$$\left(\sum_{k=j_{m-1}}^{n} |c_{(\mathbf{j},k)}(P)|^{p'}\right)^{\frac{1}{p'}} \leqslant me^{1+\frac{m-1}{p}} |[\mathbf{j}]|^{\frac{1}{p}} \|P\|_{\mathcal{P}(^m\ell_p^n)}, \tag{2.9}$$

for every $P \in \mathcal{P}(^m\mathbb{C}^n)$ as it appears in [BDS19, Lemma 3.5].

Proposition 2.1.8. *Let* $m, n \in \mathbb{N}$*,* $1 \leqslant p \leqslant \infty$ *and* $Q \in \mathcal{L}(\ell_p^n, \mathcal{P}(^{m-1}\ell_p^n))$ *be the linear operator given by*

$$Q(w)(z) := \sum_{\mathbf{j}\in\mathcal{J}(m,n)} \left(\sum_{k=1}^{n} b_{(\mathbf{j},k)} w_k\right) z_{\mathbf{j}},$$

for $z, w \in \mathbb{C}^n$*. Then for any* $\mathbf{j} \in \mathcal{J}(m,n)$ *and* $\alpha = \alpha(\mathbf{j}) \in \Lambda(m-1,n)$ *it holds*

$$\left(\sum_{k=1}^{n} |b_{(\mathbf{j},k)}|^{p'}\right)^{1/p'} \leqslant \left(\frac{(m-1)^{m-1}}{\alpha^{\alpha}}\right)^{1/p} \|Q\|.$$

Proof. For every $w \in B_{\ell_p^n}$ we may consider $P_w = Q(w) \in \mathcal{P}(^{m-1}\ell_p^n)$. Notice that for $\mathbf{j} \in \mathcal{J}(m-1,n)$ and $\alpha = \alpha(\mathbf{j})$ we have $a_\alpha(P_w) = \sum_{k=1}^n b_{(\mathbf{j},k)} w_k$. By Lemma 2.1.4 it holds

$$\begin{aligned}\left|\sum_{k=1}^{n} b_{(\mathbf{j},k)} w_k\right| = |a_\alpha(P_w)| &\leqslant \left(\frac{(m-1)^{m-1}}{\alpha^{\alpha}}\right)^{1/p} \|P_w\|_{\mathcal{P}(^m\ell_p^n)} \\ &= \left(\frac{(m-1)^{m-1}}{\alpha^{\alpha}}\right)^{1/p} \|Q(w)\|_{\mathcal{P}(^m\ell_p^n)} \leqslant \frac{(m-1)^{m-1}}{\alpha^{\alpha}} \|Q\|.\end{aligned}$$

Taking the supremum over $w \in B_{\ell_p^n}$ and using the duality $(\ell_p)' = \ell_{p'}$ the result follows. ☐

A last remark will be needed to prove Theorem 2.1.7. Notice that, fixed $m, n \in \mathbb{N}$, given $\mathbf{j} \in \mathcal{J}(m-1,n)$ and $1 \leqslant k \leqslant n$ the cardinal of the equivalence class of $(\mathbf{j}, k)$ compared to $|[\mathbf{j}]|$ meets the inequality

$$|[(\mathbf{j}, k)]| \leqslant m|[\mathbf{j}]|. \tag{2.10}$$

This holds because the case in which $|[(\mathbf{j}, k)]|$ is the largest possible is achieved when $k \neq j_i$ for every $1 \leqslant i \leqslant m-1$. Also, in that case the number of different vectors that are obtained mixing $(\mathbf{j}, k)$ is $m \times |[\mathbf{j}]|$, as k generates a different vector for each position.

Proof of Theorem 2.1.7. Given $P \in \mathcal{P}(^m\mathbb{C}^n)$ take $T \in \mathcal{L}(^m\mathbb{C}^n)$ its associated symmetric m-linear form. By Remark 1.2.5 for every $z^{(1)}, \ldots, z^{(m)}\mathbb{C}^n$ we can write

$$T(z^{(1)}, \ldots, z^{(m)}) = \sum_{\mathbf{i} \in \mathcal{M}(m,n)} c_{\mathbf{i}}(T) z^{(1)}_{i_1} \cdots z^{(m)}_{i_m},$$

with $c_{\mathbf{i}}(T) = \frac{c_{\mathbf{j}}(P)}{|\mathbf{j}|}$ if $\mathbf{i} \in [\mathbf{j}]$.

We may define $Q \in \mathcal{L}(\ell_p^n, \mathcal{P}(^{m-1}\ell_p^n))$ for $z, w \in \mathbb{C}^n$ by

$$Q(w)(z) := T(\underbrace{z, \ldots, z}_{m-1}, w).$$

Let us compute $Q(z)(w)$ based on the coefficients of T

$$\begin{aligned}
Q(w)(z) = T(z, \ldots, z, w) &= \sum_{\mathbf{i} \in \mathcal{M}(m,n)} c_{\mathbf{i}}(T) z_{i_1} \cdots z_{i_{m-1}} w_{i_m} \\
&= \sum_{\mathbf{i} \in \mathcal{M}(m-1,n)} \sum_{k=1}^{n} c_{(\mathbf{i},k)}(T) z_{\mathbf{i}} w_k \\
&= \sum_{\mathbf{j} \in \mathcal{J}(m-1,n)} \sum_{\mathbf{i} \in [\mathbf{j}]} \sum_{k=1}^{n} c_{(\mathbf{i},k)}(T) z_{\mathbf{i}} w_k \\
&= \sum_{\mathbf{j} \in \mathcal{J}(m-1,n)} |[\mathbf{j}]| \sum_{k=1}^{n} c_{(\mathbf{j},k)}(T) z_{\mathbf{j}} w_k \\
&= \sum_{\mathbf{j} \in \mathcal{J}(m-1,n)} \left(\sum_{k=1}^{n} |[\mathbf{j}]| c_{(\mathbf{j},k)}(T) w_k \right) z_{\mathbf{j}},
\end{aligned}$$

as for every $\mathbf{i} \in [\mathbf{j}]$ it holds $c_{(\mathbf{i},k)}(T) = c_{(\mathbf{j},k)}(T)$.

Using Proposition 2.1.8, the inequality in (2.10), and Harris inequality in Theorem 1.2.7 it follows

$$\begin{aligned}\left(\sum_{k=j_{m-1}}^{n}(c_{(\mathbf{j},k)}(P))^{p'}\right)^{1/p'} &= \left(\sum_{k=1}^{n}(|[(\mathbf{j},k)]|c_{(\mathbf{j},k)}(T))^{p'}\right)^{1/p'}\\ &\leqslant m\left(\sum_{k=1}^{n}(|[\mathbf{j}]|c_{(\mathbf{j},k)}(T))^{p'}\right)^{1/p'}\\ &\leqslant m\left(\frac{(m-1)^{m-1}}{\alpha^{\alpha}}\right)^{1/p}\|Q\| \leqslant m\left(\frac{(m-1)^{m-1}}{\alpha^{\alpha}}\right)^{1/p} e\|P\|,\end{aligned}$$

where $\alpha = \alpha(\mathbf{j})$. □

One last summability inequality that will be need in the following chapters is given by the next theorem.

Theorem 2.1.9. *Let $m, n \in \mathbb{N}$ and $P \in \mathcal{P}(^m\mathbb{C}^n)$ it holds*

$$\sum_{k=1}^{n}\left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,k)}|c_{(\mathbf{j},k)}(P)|^2\right)^{\frac{1}{2}} \leqslant em2^{\frac{m-1}{2}}\|P\|_{\mathcal{P}(^m\ell_\infty^n)}.$$

Theorem 2.1.9 and its proof can be found in [BDF$^+$17, Lemma 2.5] in a more general version.

Notice that inequalities in Theorem 2.1.7 and Theorem 2.1.9 may be rewritten in terms of comparing two norms in the space of m-homogeneous polynomials in n complex variables. If we consider the mixed norm of the coefficients on $\mathcal{P}(^m\mathbb{C}^n)$ defined at some polynomial P by

$$|P|_{(\infty,\dots,\infty,p')} := \sup_{\mathbf{j}\in\mathcal{J}(m-1,n)}\left(\sum_{k=j_{m-1}}^{n}(c_{(\mathbf{j},k)}(P))^{p'}\right)^{1/p'},$$

we may write Theorem 2.1.7 in this terms as

$$|P|_{(\infty,\dots,\infty,p')} \leqslant em(m-1)^{m-1}\|P\|_{\ell_p^n}, \tag{2.11}$$

for every $1 \leqslant p \leqslant \infty$, $m, n \in \mathbb{N}$ and $P \in \mathcal{P}(^m\mathbb{C}^n)$.

On the other hand we may define for $P(z) = \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}}(P)z_{\mathbf{j}} \in \mathcal{P}(^m\mathbb{C}^n)$ the norm

$$|P|_{(1,2,\dots,2)} := \sum_{k=1}^{n}\left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,k)}|c_{(\mathbf{j},k)}(P)|^2\right)^{\frac{1}{2}},$$

now we can translate Theorem 2.1.9 in this terms as

$$|P|_{(1,2,\dots,2)} \leqslant em2^{\frac{m-1}{2}}\|P\|_{\ell_\infty^n}, \tag{2.12}$$

for every $m, n \in \mathbb{N}$ and $P \in \mathcal{P}(^m\mathbb{C}^n)$.

In both cases this gives a sort of Hardy-Littlelwood inequality, especially interesting for $1 \leqslant p \leqslant m$ where we have shown classical Hardy-Littlelwood inequalities doesn't hold for $q < \infty$. For $1 \leqslant p < m$ (2.11) gives a summability inequality that is better than (2.7). On the other hand in comparing the classical Bonhenblust-Hille inequality in Theorem 2.1.2 with (2.12), this last result gives a bound for a mixed coefficient norm where the exponents do not depend on m, this will be crucial on the study of holomorphic functions.

2.2 Beyond summability

If we change any of the parameters involved on either or both sides of the Hardy-Littlewood inequalities in (2.4) beyond the limits described in Remark 2.1.5 and Theorem 2.1.6, it is expected that the dependence on the number of variables becomes apparent. It is worth asking how this reliance is in terms of the summability of the coefficients, the uniform norm and the homogeneity degree considered.
Analogously, we can study a similar problem: the inequality that comes from exchanging the roles (sides of the inequality) between the norm of the coefficients and the uniform norm.

Problem 2.2.1. *Let $A^m_{p,q}(n)$ and $B^m_{q,p}(n)$ be the smallest constants that fulfill the following inequalities: for every m-homogeneous polynomial P in n complex variables,*

$$|P|_q \leqslant A^m_{p,q}(n)\ \|P\|_{\mathcal{P}(^m\ell_p^n)},$$
$$\|P\|_{\mathcal{P}(^m\ell_p^n)} \leqslant B^m_{q,p}(n)\ |P|_q.$$

How these constants behave in terms of the number of variables n? Which is their exact asymptotic growth?

From the operator theoretic point of view these constants are exactly the norm of the identity between the Banach spaces $\mathcal{P}(^m\ell_p^n)$ and $(\mathcal{P}(^m\mathbb{C}^n), |\cdot|_q)$, i.e.,

$$\begin{aligned} A^m_{p,q}(n) &= \|id : \mathcal{P}(^m\ell_p^n) \to (\mathcal{P}(^m\mathbb{C}^n), |\cdot|_q)\ \|, \\ B^m_{q,p}(n) &= \|id : (\mathcal{P}(^m\mathbb{C}^n), |\cdot|_q) \to \mathcal{P}(^m\ell_p^n)\|. \end{aligned} \tag{2.13}$$

Observe that by (2.3), the dependence on n of the constant that appears when comparing the sup-norm with the Bombieri norm is exactly the same as the constants related to Problem 2.2.1.

In the 80's, Goldberg [Gol87] settled a similar problem in the context of matrix theory: given an $n \times n$ matrix A, he was interested in finding the best equivalence constant $c(q, p, n)$ (or its asymptotic behavior as n tends to infinity) which relates the ℓ_q-norm of the coefficients with the operator norm of A acting on ℓ_p^n. Partial and sharp results of this problem (and also some variants of it) were given by Feng and Tonge in [Ton00, Fen03, FT07]. Observe that Problem 2.2.1 is essentially a polynomial version of Golberg's problem.

To attack the main problem of this section it will be useful to compare, given a polynomial $P \in \mathcal{P}(^m\mathbb{C}^n)$ and $1 \leqslant r, s \leqslant \infty$, the uniform norms $\|P\|_{\mathcal{P}(^m\ell_r^n)}$ and $\|P\|_{\mathcal{P}(^m\ell_s^n)}$, and the coefficient norms $|P|_r$ and $|P|_s$. Recall that, for every $1 \leqslant r \leqslant s \leqslant \infty$ and $z \in \mathbb{C}^n$, the following relation between the norms in ℓ_r^n and ℓ_s^n hold

$$\|z\|_s \leqslant \|z\|_r \leqslant n^{\frac{1}{r}-\frac{1}{s}} \|z\|_s, \tag{2.14}$$

which may be rephrased in terms of the balls of both spaces as

$$B_{\ell_r^n} \subset B_{\ell_s^n} \subset n^{\frac{1}{r}-\frac{1}{s}} B_{\ell_r^n}. \tag{2.15}$$

We will need in many opportunities the *Stirling formula* or *Stirling inequality*, which asserts that

$$\sqrt{2\pi k}\left(\frac{k}{e}\right)^k e^{\frac{1}{12k+1}} \leqslant k! \leqslant \sqrt{2\pi k}\left(\frac{k}{e}\right)^k e^{\frac{1}{12k}}, \tag{2.16}$$

for every positive integer k. Using this formula we get

$$\begin{aligned}\binom{m+n-1}{m} = \frac{(m+n-1)!}{(n-1)!m!} &\leqslant \sqrt{\frac{m+n-1}{(n-1)m}} \frac{(m+n-1)^{m+n-1}}{m^m (n-1)^{n-1}} \\ &\leqslant 2\left(1+\frac{n-1}{m}\right)^m \left(1+\frac{m}{n-1}\right)^{n-1} \leqslant 2e^m n^m.\end{aligned} \tag{2.17}$$

Remark 2.2.2. Given $1 \leqslant r \leqslant s \leqslant \infty$ and $m, n \in \mathbb{N}$, for every $P \in \mathcal{P}(^m\mathbb{C}^n)$ it holds

$$|P|_s \leqslant |P|_r \leqslant \binom{m+n-1}{n}^{\frac{1}{r}-\frac{1}{s}} |P|_s \leqslant (2e^m)^{\frac{1}{r}-\frac{1}{s}} n^{m(\frac{1}{r}-\frac{1}{s})} |P|_s, \tag{2.18}$$

and

$$\|P\|_{\mathcal{P}(^m\ell_r^n)} \leqslant \|P\|_{\mathcal{P}(^m\ell_s^n)} \leqslant n^{m(\frac{1}{r}-\frac{1}{s})} \|P\|_{\mathcal{P}(^m\ell_r^n)}. \tag{2.19}$$

Proof. Fix $P \in \mathcal{P}(^m\mathbb{C}^n)$. Let us begin with the coefficient norm comparison in (2.18). Remember the dimension of $\mathcal{P}(^m\mathbb{C}^n)$ as a $\mathbb{C}$ vector space is $\binom{m+n-1}{m}$. Then by the definition of the coefficient norm it is plain $(\mathcal{P}(^m\mathbb{C}^n), |\cdot|_r)$ is isometric to $\ell_r^{\binom{m+n-1}{m}}$, so by (2.14) and using the bound in (2.17) we have

$$|P|_s \leqslant |P|_r \leqslant \binom{m+n-1}{n}^{\frac{1}{r}-\frac{1}{s}} |P|_s \leqslant (2e^m)^{\frac{1}{r}-\frac{1}{s}} n^{m(\frac{1}{r}-\frac{1}{s})} |P|_s.$$

For the comparison in (2.19) we use (2.15) and the homogeneity of P, then

$$\|P\|_{\mathcal{P}(^m\ell_r^n)} = \|P\|_{B_{\ell_r^n}} \leqslant \|P\|_{B_{\ell_s^n}} \leqslant \|P\|_{n^{\frac{1}{r}-\frac{1}{q}} B_{\ell_s^n}} = n^{m(\frac{1}{r}-\frac{1}{q})} \|P\|_{B_{\ell_r^n}}. \qquad \square$$

2.2.1 Random polynomials

As usual, to understand the norm of bounded linear operators it is necessary to have a global bound for every element on the domain and frequently some particular element to show the sharpness of the bound. Now we present a family of polynomials that will show the sharpness on the bounds in many different contexts through this thesis. The existence of these polynomial is proved using probabilistic methods.

Bayart in [Bay12] (see also [Boa00, DGM03, DGM04]) exhibited polynomials with unimodular coefficients and with small sup-norm on the unit ball of ℓ_p^n. Moreover, he showed that for each $1 < p \leqslant \infty$ and every coefficient sequence $(a_\alpha)_{\alpha\in\Lambda(m,n)}$ there exists a sequence of signs $(\varepsilon_\alpha)_{\alpha\in\Lambda(m,n)} \subset \mathbb{T}$ which defines an m-homogeneous polynomial $P(z) := \sum_{\alpha\in\Lambda(m,n)} \varepsilon_\alpha a_\alpha z^\alpha$ in n complex variables such that

$$\|P\|_{\mathcal{P}(^m\ell_p^n)} \leqslant K_{m,p} \times \begin{cases} n^{1-\frac{1}{p}} & \text{if } 1 < p \leqslant 2, \\ n^{m(\frac{1}{2}-\frac{1}{p})+\frac{1}{2}} & \text{if } 2 \leqslant p \leqslant \infty, \end{cases} \tag{2.20}$$

where $K_{m,p} \leqslant C\log(m)^{1-\frac{1}{p}} \sup_{\alpha\in\Lambda(m,n)} \left\{ |a_\alpha| \left(\frac{\alpha!}{m!}\right)^{1/p} \right\}$ for some $C > 0$ independent of n, m and p.

We may choose $a_\alpha = 1$ for every $\alpha \in \Lambda(m,n)$. In this case we get an unimodular polynomial and

$$K_{m,p} \leqslant C\log(m)^{1-\frac{1}{p}}. \tag{2.21}$$

Also, the number of non-zero coefficients is exactly the number of possible monomials, $\binom{n+m-1}{m}$. These polynomials will be very useful: they will be extremal in a wide ranges of values of $p, r \in [1,\infty]$ for the first inequality of Problem 2.2.1. Unfortunately, for a large range of values of p and r these polynomials become useless and new extremal examples are needed. Therefore, it is important to relax the number of terms appearing in the polynomials, by allowing them to have some zero coefficients, in order to reduce quantitatively the value of the sup-norm. Obviously if one gets rid of many coefficients/monomials this helps considerably to lower the value of the norm but the important thing is to maintain an appropriate balance (having a sufficient number of non-zero coefficients but keeping the norm small).

We introduce the so-called *Steiner polynomials*, a special class of tetrahedral polynomials defined by Dixon in [Dix76] and studied there with uniform norm in ℓ_∞^n. In [GMSP15] the authors analyze the case of the tetrahedral polynomial with uniform norm in ℓ_p^n. These polynomial turn out to give accurate enough lower bounds for the constant $A_{p,q}^m(n)$ in many cases.

We need some definitions to describe them. An $S_p(t,m,n)$ partial Steiner system is a collection of subsets of size m of $\{1,\ldots,n\}$ such that every subset of t elements is contained in at most one member of the collection of subsets of size m. Notice that we may see every $S_p(t,m,n)$ partial Steiner system $\mathcal{S}$ as a subset of the index set $\mathcal{J}(m,n)$. An m-homogeneous polynomial P of n variables is a Steiner polynomial if there exists an $S_p(t,m,n)$ partial Steiner system $\mathcal{S}$ such that $P(z_1,\ldots,z_n) = \sum_{\mathbf{j}\in\mathcal{S}} c_{\mathbf{j}} z_{\mathbf{j}}$ and $c_{\mathbf{j}} = \pm 1$. Note that the monomials involved in this class have a particular combinatorial configuration.

The following result appears in [GMSP15, Theorem 2.5.].

Theorem 2.2.3. *Let $m \geqslant 2$ and $\mathcal{S}$ be an $S_p(m-1,m,n)$ partial Steiner system. Then there exist signs $(c_{\mathbf{j}})_{\mathbf{j}\in\mathcal{S}}$ and a constant $D_{m,p} > 0$ independent of n such that the m-homogeneous polynomial $P = \sum_{\mathbf{j}\in\mathcal{S}} c_{\mathbf{j}} z_{\mathbf{j}}$ satisfies*

$$\|P\|_{\mathcal{P}(^m\ell_p^n)} \leqslant D_{m,p} \times \begin{cases} \log^{\frac{3p-3}{p}}(n) & \text{for } 1 \leqslant p \leqslant 2, \\ \log^{\frac{3}{p}}(n) n^{m(\frac{1}{2}-\frac{1}{p})} & \text{for } 2 \leqslant p < \infty. \end{cases}$$

Moreover, the constant $D_{m,p}$ may be taken independent of m for $p \neq 2$.

The last ingredient we need for the applications is the existence of nearly optimal partial Steiner systems, in the sense that they have many elements. This translates to many unimodular coefficients of the Steiner polynomials. It is well known that any partial Steiner system $S_p(m-1,m,n)$ has cardinality less than or equal to $\frac{1}{m}\binom{n}{m-1}$.

Rödl [Röd85] in the eighties proved that there exist partial Steiner systems $S_p(m-1,m,n)$ of cardinality at least $(1-o(1))\frac{1}{m}\binom{n}{m-1}$, where $o(1)$ tends to zero as n goes to infinity. Taking partial Steiner systems of this cardinality in Theorem 2.2.3 we have the following.

Corollary 2.2.4. *Let $m \geqslant 2$. Then there exists a m-homogeneous Steiner unimodular polynomial P of n complex variables with at least $C_m n^{m-1}$ unimodular coefficients satisfying the estimates in Theorem 2.2.3, where C_m is a constant that depends only on m.*

2.2.2 A partial solution to the problem.

If $(a_n)_n$ and $(b_n)_n$ are two sequences of real numbers we will write $a_n \ll b_n$ if there exists a constant $C > 0$ (independent of n) such that $a_n \leqslant C b_n$ for every n. We will write $a_n \sim b_n$ if $a_n \ll b_n$ and $b_n \ll a_n$. Recall that the number of m-homogeneous monomials in n variables is $|\mathcal{J}(m,n)| = \binom{n+m-1}{m} \sim n^m$.

Given $T \in \mathcal{L}(^m\mathbb{C}^n)$ we will denote its r-th coefficients norm by $|T|_r$, that is,

$$|T|_r := \Big(\sum_{i\in\mathcal{M}(m,n)} |T(e_{i_1},\ldots,e_{i_m})|^r \Big)^{\frac{1}{r}},$$

where $\mathcal{M}(m,n) = \{\mathbf{i} = (i_1,\ldots,i_m) : 1 \leqslant i_l \leqslant n, 1 \leqslant l \leqslant m\}$. Using Remark 1.2.5 and Harris formula on Theorem 1.2.7 it is no hard to see there exist constants $C_l = C_l(m) > 0$, $l = 1,2$, independent of n, such that for every $P \in \mathcal{P}(^m\mathbb{C}^n)$ and its associated symmetric m-linear form $\check{P}$ we have

$$|\check{P}|_r \leqslant |P|_r \leqslant C_1 |\check{P}|_r \qquad \text{for } 1 \leqslant r \leqslant \infty, \tag{2.22}$$

$$C_2 \|\check{P}\|_{\mathcal{L}(^m\ell_p^n)} \leqslant \|P\|_{\mathcal{P}(^m\ell_p^n)} \leqslant \|\check{P}\|_{\mathcal{L}(^m\ell_p^n)} \qquad \text{for } 1 \leqslant p \leqslant \infty. \tag{2.23}$$

It will be useful to notice that, as $(\mathcal{P}(^m\mathbb{C}^n), |\cdot|_r)$ is an $L_p(U,\mu)$ space where μ is the counting measure and $U = \Lambda(m,n)$, then by equation (1.13) we have

$$[(\mathcal{P}(^m\mathbb{C}^n), |\cdot|_{r_0}), (\mathcal{P}(^m\mathbb{C}^n), |\cdot|_{r_1})]_\theta = (\mathcal{P}(^m\mathbb{C}^n), |\cdot|_r) \tag{2.24}$$

for every $0 < \theta < 1$ where $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$. We will need this in the proof of the following theorem but it also will be needed all along the thesis.

We now state our main theorem of this chapter.

Theorem 2.2.5. *Let $A^m_{p,q}(n)$ be the smallest constant such that, for every m-homogeneous polynomial P in n complex variables, $|P|_q \leqslant A^m_{p,q}(n) \, \|P\|_{\mathcal{P}(^m\ell_p^n)}$. Then,*

$$\begin{cases} A^m_{p,q}(n) \sim 1 & \text{for } (A): [\frac{1}{2} \leqslant \frac{1}{q} \leqslant \frac{m+1}{2m} - \frac{1}{p}] \text{ or } [\frac{1}{q} \leqslant \frac{1}{2} \ \wedge \ \frac{m}{p} \leqslant 1 - \frac{1}{q}], \\ A^m_{p,q}(n) \sim n^{\frac{m}{p}+\frac{1}{q}-1} & \text{for } (B): [\frac{1}{2m} \leqslant \frac{1}{p} \leqslant \frac{1}{m} \ \wedge \ -\frac{m}{p}+1 \leqslant \frac{1}{q}], \\ A^m_{p,q}(n) \sim n^{m(\frac{1}{p}+\frac{1}{q}-\frac{1}{2})-\frac{1}{2}} & \text{for } (C): [\frac{m+1}{2m} \leqslant \frac{1}{q} \ \wedge \ \frac{1}{p} \leqslant \frac{1}{2}] \text{ or} \\ & \qquad [\frac{1}{2} \leqslant \frac{1}{q} \leqslant \frac{m+1}{2m} \leqslant \frac{1}{p} + \frac{1}{q} \wedge \ \frac{1}{p} \leqslant \frac{1}{2}], \\ A^m_{p,q}(n) \sim n^{\frac{m}{q}+\frac{1}{p}-1} & \text{for } (D): [\frac{1}{2} \leqslant \frac{1}{p} \ \wedge \ 1 - \frac{1}{p} \leqslant \frac{1}{q}], \\ A^m_{p,q}(n) \ll n^{\frac{m-1}{q}} & \text{for } (E): [\frac{1}{2} \leqslant \frac{1}{p} \leqslant 1 - \frac{1}{q}], \\ A^m_{p,q}(n) \sim n^{\frac{1}{q}} & \text{for } (F): [\frac{m-1}{p} \leqslant 1 - \frac{1}{q} \ \wedge \ \frac{1}{m} \leqslant \frac{1}{p} \leqslant \frac{1}{m-1}], \end{cases}$$

Moreover, the power of n in (E) cannot be improved.

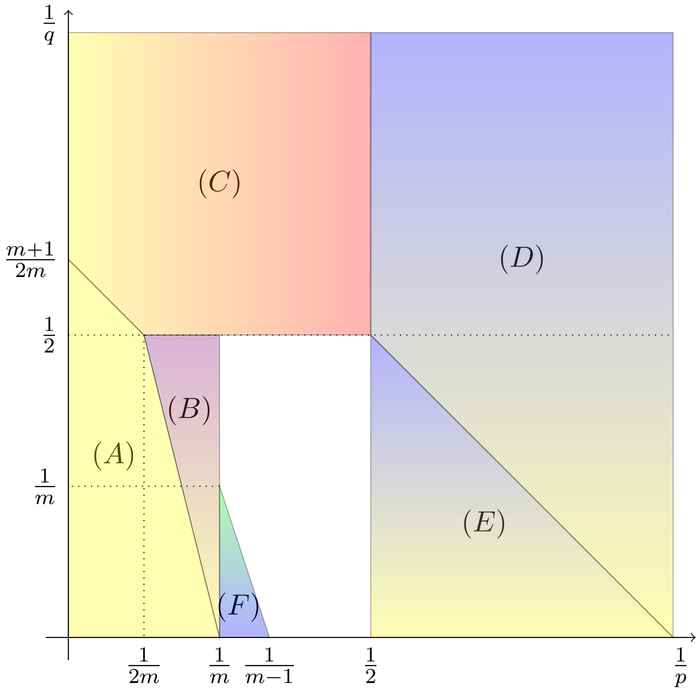

Figure 2.1: Graphical overview of the regions described in Theorem 2.2.5.

Figure 2.1 represents the regions described in Theorem 2.2.5. For the blank region we do not know right order of $A^m_{p,q}(n)$ (see the comments after Corollary 2.3.4 below). It is noteworthy that much of the work is to determine *which are the regions* to consider.

Also note that for $m = 2$ we have a complete description of the asymptotics of $A^2_{p,q}(n)$. For $q \geqslant 2$ this can also be deduced as a consequence of [FT07, Theorems 1 and 2].

Proof. Let P be an m-homogeneous polynomial in n complex variables and T its associated symmetric m-linear form.

$\bullet(A)$: Suppose first that $\frac{1}{2} \leqslant \frac{1}{q} \leqslant \frac{m+1}{2m} - \frac{1}{p}$. If $r := \frac{2mq}{(m+1)q-2m}$ then $2m \leqslant r \leqslant p$ and by the Hardy-Littlewood inequality, Theorem 2.1.6 (ii), we have

$$|P|_q \ll \|P\|_{\mathcal{P}(^m\ell_r^n)} \ll \|P\|_{\mathcal{P}(^m\ell_p^n)}.$$

Now suppose $\frac{1}{q} \leqslant \frac{1}{2}$ and $\frac{m}{p} \leqslant 1 - \frac{1}{q}$. If we set $r := \frac{mq}{q-1}$ then $m \leqslant r \leqslant \min\{p, 2m\}$; then reasoning as before (but using part (i) of Theorem 2.1.6) we can easily reach the same conclusion.

$\bullet(B)$: Taking $p \leqslant r = \frac{mq}{q-1}$, by the Hardy-Littlewood inequality, Theorem 2.1.6 (i), and (2.19) it follows

$$|P|_q \ll \|P\|_{\mathcal{P}(^m\ell_r^n)} \ll \|P\|_{\mathcal{P}(^m\ell_p^n)} n^{m(\frac{1}{p}-\frac{1}{r})} = \|P\|_{\mathcal{P}(^m\ell_p^n)} n^{\frac{m}{p}+\frac{1}{q}-1}.$$

For the optimality we can take the polynomial

$$P = \sum_{j=0}^{k-1} z_{mj+1} \cdots z_{mj+m}, \quad \text{with} \quad k = \left[\frac{n}{m}\right],$$

it can be seen using Lagrange multipliers and the fact that $p \geqslant m$, that

$$\|P\|_{\mathcal{P}(^m\ell_p^n)} = k\Big(\frac{1}{mk}\Big)^{\frac{m}{p}} \sim n^{1-\frac{m}{p}}.$$

Then,

$$n^{\frac{1}{q}} \sim k^{\frac{1}{q}} = |P|_q \leqslant A^m_{p,q}(n) \|P\|_{\mathcal{P}(^m\ell_p^n)} \sim A^m_{p,q}(n) n^{1-\frac{m}{p}},$$

and therefore $n^{\frac{m}{p}+\frac{1}{q}-1} \ll A^m_{p,q}$.

$\bullet(C)$: Suppose $\frac{m+1}{2m} \leqslant \frac{1}{q}$ and $\frac{1}{p} \leqslant \frac{1}{2}$. Using the Bohnenblust-Hille inequality Theorem 2.1.6, inequalities (2.18) and (2.19) we have

$$\begin{aligned} |P|_q &\ll n^{m(\frac{1}{q}-\frac{m+1}{2m})} |P|_{\frac{2m}{m+1}} \ll n^{m(\frac{1}{q}-\frac{m+1}{2m})} \|P\|_{\mathcal{P}(^m\ell_\infty^n)} \\ &\ll n^{m(\frac{1}{q}-\frac{m+1}{2m})} n^{\frac{m}{p}} \|P\|_{\mathcal{P}(^m\ell_p^n)} = n^{m(\frac{1}{p}+\frac{1}{q}-\frac{1}{2})-\frac{1}{2}} \|P\|_{\mathcal{P}(^m\ell_p^n)}. \end{aligned}$$

Suppose $\frac{1}{2} \leqslant \frac{1}{q} \leqslant \frac{m+1}{2m} \leqslant \frac{1}{p} + \frac{1}{q}$ and let $r := \frac{2mq}{(m+1)q-2m}$. Note that $\max\{2m, p\} \leqslant r$. By the Hardy-Littlewood inequality, Theorem 2.1.6 (ii) and (2.19) we get

$$|P|_q \ll \|P\|_{\mathcal{P}(^m\ell_r^n)} \ll n^{m(\frac{1}{p}-\frac{1}{r})} \|P\|_{\mathcal{P}(^m\ell_p^n)} = n^{m(\frac{1}{p}+\frac{1}{q}-\frac{1}{2})-\frac{1}{2}} \|P\|_{\mathcal{P}(^m\ell_p^n)}.$$

To show that this asymptotic growth is optimal, we consider P an m-homogeneous unimodular polynomial as in (2.20). Then, as $\frac{1}{p} \leqslant \frac{1}{2}$

$$n^{\frac{m}{q}} \ll |P|_q \leqslant A^m_{p,q}(n)\|P\|_{\mathcal{P}(^m\ell_p^n)} \ll A^m_{p,q}(n)n^{m(\frac{1}{2}-\frac{1}{p})+\frac{1}{2}}.$$

Therefore,

$$n^{m(\frac{1}{p}+\frac{1}{q}-\frac{1}{2})-\frac{1}{2}} = n^{\frac{m}{q}-[m(\frac{1}{2}-\frac{1}{p})+\frac{1}{2}]} \ll A^m_{p,q}(n).$$

$\bullet(D)$: If T is the symmetric m-linear form associated to P, it induces a $(m-1)$-linear mapping $\tilde{T} \in \mathcal{L}(^{m-1}l_p^n;(l_p^n)^*)$, defined by

$$\tilde{T}(x_1,\ldots,x_{m-1})(\cdot) = T(x_1,\ldots,x_{m-1},\cdot).$$

Then

$$\begin{aligned}
|T|_q^q &= \sum_{i\in\mathcal{M}(m,n)} |T(e_{i_1},\ldots,e_{i_m})|^q \\
&= \sum_{i\in\mathcal{M}(m-1,n)} \sum_{l=1}^n |T(e_{i_1},\ldots,e_{i_{m-1}},e_l)|^q \\
&\leqslant \sum_{i\in\mathcal{M}(m-1,n)} (\sum_{l=1}^n |T(e_{i_1},\ldots,e_{i_{m-1}},e_l)|^{p'})^{\frac{q}{p'}} n^{\frac{q}{p}+1-q} \\
&\leqslant n^{m-1+\frac{r}{p}+1-q} \sup_{\|x_i\|_p\leqslant 1} \|\tilde{T}(x_1,\ldots,x_{m-1})\|_{p'}^q \\
&= n^{m+\frac{q}{p}-q}\|T\|^q_{\mathcal{L}(^m\ell_p^n)},
\end{aligned}$$

where in the first inequality we used Hölder inequality in the case $\frac{p'}{q} \geqslant 1$. Then by equations (2.22) and (2.23) we have

$$|P|_q \ll n^{\frac{m}{q}+\frac{1}{p}-1}\|P\|_{\ell_p^n}.$$

For the optimality, we use (2.20) so, since $1 \leqslant p \leqslant 2$ there exists a unimodular polynomial P such that

$$n^{\frac{m}{r}} \ll |P|_r \leqslant A^m_{p,q}(n)\|P\|_{\mathcal{P}(\ell_p^n)} \ll A^m_{p,q}(n)n^{1-\frac{1}{p}}.$$

$\bullet(E)$: Observe that

$$A^m_{p,q}(n) = \|id : \mathcal{P}(^m\ell_p^n) \to (\mathcal{P}(^m\mathbb{C}^n), |\cdot|_q)\|.$$

Thus, if $\frac{1}{q} = \frac{\theta}{p'}$, for $0 < \theta < 1$, using the multilinear interpolation Theorem 1.4.1 we conclude that

$$A^m_{p,q}(n) \leqslant (A^m_{p,p'}(n))^\theta (A^m_{p,\infty}(n))^{1-\theta}.$$

Since $1 \leqslant p \leqslant 2$, we have by part (D) that $A^m_{p,p'}(n) \sim n^{\frac{m-1}{p'}}$ and also, applying the Cauchy integral formula we deduce that $A^m_{p,\infty}(n) \sim 1$. Therefore,

$$\begin{aligned}
&\|id : \mathcal{P}(^m\ell_p^n) \to (\mathcal{P}(^m\mathbb{C}^n), |\cdot|_{p'})\| \ll n^{\frac{m-1}{p'}}, \\
&\|id : \mathcal{P}(^m\ell_p^n) \to (\mathcal{P}(^m\mathbb{C}^n), |\cdot|_{\infty})\| \ll 1
\end{aligned}$$

by Theorem 1.4.1 and (2.24) we obtain

$$\|id : \mathcal{P}(^m\ell_p^n) \to (\mathcal{P}(^m\mathbb{C}^n), |\cdot|_q)\| = A_{p,q}^m(n) \leqslant A_{p,q}^m(n) \leqslant (A_{p,p'}^m(n))^\theta (A_{p,\infty}^m(n))^{1-\theta} \ll n^{\frac{m-1}{q}}.$$

For the lower bound, taking a Steiner polynomial $P \in \mathcal{P}(^m\mathbb{C}^n)$ as in Corollary 2.2.4 whose associated partial Steiner system has cardinality $\gg n^{m-1}$ and $1 \leqslant p \leqslant 2$, then

$$n^{\frac{m-1}{q}} \ll |P|_q \leqslant A_{p,q}^m(n)\|P\|_{\mathcal{P}(^m\ell_p^n)} \ll A_{p,q}^m(n)\log^{\frac{3p-3}{p}}(n).$$

Hence, we have that for every $\varepsilon > 0$,

$$n^{\frac{m-1}{q}-\varepsilon} \ll A_{p,q}^m(n).$$

$\bullet(F)$: Let T be the symmetric m-linear form associated to P and, given $1 \leqslant i \leqslant n$, let us define $T_i \in \mathcal{L}(^{m-1}\mathbb{C}^n)$ as

$$T_i(x_2, \ldots, x_m) = T(e_i, x_2, \ldots, x_m).$$

Then

$$\begin{aligned} |P|_q^q \sim |T|_q^q &= \sum_{i \in \mathcal{M}(m,n)} |T(e_{i_1}, \ldots, e_{i_m})|^q \\ &= \sum_{i=1}^n |T_i|_q^q \\ &\ll \sum_{i=1}^n \|T_i\|_{\mathcal{L}(^{m-1}\ell_p^n)}^q \qquad (2.25) \\ &\leqslant n\|T\|_{\mathcal{L}(^m\ell_p^n)}^q \sim n\|P\|_{\mathcal{P}(^m\ell_p^n)}^q, \end{aligned}$$

where we have used in (2.25) the fact that $A_{p,q}^{m-1}(n) \sim 1$ for this range of p and q. Therefore

$$|P|_q \ll n^{\frac{1}{q}}\|P\|_{\mathcal{P}(^m\ell_p^n)}.$$

For the lower bound, let $P = \sum_{j=1}^k z_{mj+1} \cdots z_{mj+m}$ as in part (B), then since $p \geqslant m$ (in region (F)), we have that $\|P\|_{\mathcal{P}(^m\ell_p^n)} \sim 1$ and thus

$$n^{\frac{1}{q}} \sim |P|_q \ll A_{p,q}^m(n)\|P\|_{\mathcal{P}(^m\ell_p^n)} \sim A_{p,q}^m(n).$$ □

For $2 \leqslant p \leqslant m$, $2 \leqslant q < \infty$ and $(\frac{1}{p}, \frac{1}{q}) \notin (F)$ we could have used interpolation (in vertical direction, as we did in the proof of part (E) of Theorem 2.2.5) to obtain effective upper bounds for $A_{p,q}^m$. We choose not to state them explicitly since we believe these estimates are suboptimal.

2.2.3 Asymptotic estimates for $B^m_{r,p}(n)$

We now present the correct asymptotic behavior for the constants $B^m_{r,p}(n)$ defined in Problem 2.2.1. These estimates will be useful in the next section for the applications.

Proposition 2.2.6. *Let $B^m_{r,p}(n)$ be the smallest constant such that for every m-homogeneous polynomial P in n complex variables, $\|P\|_{\mathcal{P}(^m\ell_p^n)} \leqslant B^m_{r,p}(n)\ |P|_r$. We have*

$$B^m_{r,p}(n) \sim \begin{cases} 1 & \text{for } r \leqslant p', \\ n^{m(1-\frac{1}{p}-\frac{1}{r})} & \text{for } r \geqslant p'. \end{cases}$$

Proof. Let $n, m \in \mathbb{N}$ and $1 \leqslant p, r \leqslant \infty$. Let $P = \sum_{\alpha\in\Lambda(m,n)} a_\alpha z^\alpha$ be an m-homogeneous polynomial in n variables. Suppose first that $r \leqslant p'$. Then

$$\begin{aligned} \|P\|_{\mathcal{P}(^m\ell_p^n)} &= \sup_{z\in B_{\ell_p^n}} \Big|\sum_{\alpha\in\Lambda(m,n)} a_\alpha z^\alpha\Big| \\ &\leqslant \sup_{z\in B_{\ell_p^n}} \Big(\sum_{\alpha\in\Lambda(m,n)} |a_\alpha|^{p'}\Big)^{\frac{1}{p'}} \Big(\sum_{\alpha\in\Lambda(m,n)} |z^\alpha|^p\Big)^{\frac{1}{p}} \\ &\leqslant |P|_{p'} \sup_{z\in B_{\ell_p^n}} \Big(\sum_{\mathbf{i}\in\mathcal{M}(m,n)} |z_\mathbf{i}|^p\Big)^{\frac{1}{p}} \\ &= |P|_{p'} \sup_{z\in B_{\ell_p^n}} \Big(\sum_{k=1}^n |z_k|^p\Big)^{\frac{m}{p}} = |P|_{p'} \leqslant |P|_r. \end{aligned}$$

On the other hand, if $r \geqslant p'$,

$$\|P\|_{\mathcal{P}(^m\ell_p^n)} \leqslant |P|_{p'} \leqslant |P|_r n^{m(\frac{1}{p'}-\frac{1}{r})} = |P|_r n^{m(1-\frac{1}{p}-\frac{1}{r})}.$$

To study lower bounds, let us take the polynomial $P(z) = \sum_{\mathbf{j}\in\mathcal{J}(m,n)} z_\mathbf{j}$. Note that $|P|_r \sim n^{\frac{m}{r}}$ and

$$\begin{aligned} \|P\|_{\mathcal{P}(^m\ell_p^n)} &= \sup_{z\in B_{\ell_p^n}} \Big|\sum_{\mathbf{j}\in\mathcal{J}(m,n)} z_\mathbf{j}\Big| \\ &\geqslant \Big|\sum_{\mathbf{j}\in\mathcal{J}(m,n)} n^{-\frac{m}{p}}\Big| \qquad \text{taking } z = (\overbrace{\frac{1}{n^{\frac{1}{p}}}, \dots, \frac{1}{n^{\frac{1}{p}}}}^{m}) \\ &\sim n^{m(1-\frac{1}{p})}. \end{aligned}$$

Therefore $B^m_{r,p}(n) \gg n^{m(1-\frac{1}{r}-\frac{1}{p})}$. □

2.3 Some consequences of the results

Now we present some applications of the previous results to two problems that, at a first sight, seem to be disconnected from the previous development. The first outcome involves a problem within the theory of complex interpolation of Banach spaces, the second is about von Neumann's inequality which belongs to the theory of bounded operators on Hilbert spaces.

2.3.1 Complex interpolation on spaces of polynomials

The following is a problem within the theory of complex interpolation of spaces of homogeneous polynomials in Banach spaces. There is an extremely close relationship between the tensor products of Banach spaces and the homogeneous polynomials in these spaces that we choose not to develop in this thesis. The interested reader may translate the results in this section to analogous results for the tensor product of Banach spaces endowed with the injective norm. This can be also found in [GMMb].

Given a compatible couple of Banach spaces (X, Y) and $0 < \theta < 1$ we may consider two new Banach spaces that may or may not be isomorphic

$$[\mathcal{P}(^m X), \mathcal{P}(^m Y)]_\theta \text{ and } \mathcal{P}(^m[X, Y]_\theta).$$

Defant and Michels in [DM00] and also Kouba in [Kou91] proved remarkable results on complex interpolation of injective tensor products of Banach spaces (see also [DM03]). Those results are much more general but in particular they imply,

$$[\mathcal{P}(^2\ell_{p_0}), \mathcal{P}(^2\ell_{p_1})]_\theta = \mathcal{P}(^2[\ell_{p_0}, \ell_{p_1}]_\theta) \tag{2.26}$$

for $0 < \theta < 1$, $2 \leqslant p_0, p_1 \leqslant \infty$. Equation (2.26) must be interpreted as an equality of Banach spaces, which means that there is a bounded and invertible operator between those two spaces.

We will show that Theorem 2.2.5 implies that a similar statement does not hold for the m-homogeneous case when $m > 2$. Indeed, we will show that the following problem has a negative answer.

Problem 2.3.1. *Given $m > 2$, $2 \leqslant p_0, p_1 \leqslant \infty$ and $0 < \theta < 1$, Is there an Banach spaces isomorphism between $\mathcal{P}(^m[\ell_{p_0}, \ell_{p_1}]_\theta)$ and $[\mathcal{P}(^m\ell_{p_0}), \mathcal{P}(^m\ell_{p_1})]_\theta$?*

As we have already pointed out, for $m = 2$ the answer to the question posed in Problem 2.3.1 is affirmative. It is natural to ask this question for all possible homogeneity degrees $m \in \mathbb{N}$ just for the sake of general understanding of the spaces of homogeneous polynomials, but it is also interesting due to its possible applications. In general to understand the interpolation spaces between Banach spaces plus Theorem 1.4.1 gives a very useful device to attack several functional analysis problems.

Another variant of Problem2.3.1 is expressed in the following statement.

Problem 2.3.2. *Given $m > 2$, $2 \leqslant p_0, p_1 \leqslant \infty$ and $0 < \theta < 1$, Is there a Banach space isomorphism $T : \mathcal{P}({}^m[\ell_{p_0}, \ell_{p_1}]_\theta) \to [\mathcal{P}({}^m\ell_{p_0}), \mathcal{P}({}^m\ell_{p_1})]_\theta$, such that for every $n \in \mathbb{N}$ it induces linear isomorphism $T_n : \mathcal{P}({}^m[\ell^n_{p_0}, \ell^n_{p_1}]_\theta) \to [\mathcal{P}({}^m\ell^n_{p_0}), \mathcal{P}({}^m\ell^n_{p_1})]_\theta$?*

Clearly a negative answer to Problem 2.3.1 gives a negative answer to Problem 2.3.2 as there are more requirements to the isomorphism in the second problem. We will not give a proper treatment here but the interested reader may look at the article [BM19] where the authors prove that Problem 2.3.1 and Problem 2.3.2 are actually equivalent. There Bayart and Mastylo investigate the interpolation between Banach spaces in a more general way and give some applications to the polynomial problem.

A positive answer to the question posed in Problem 2.3.2 would give us a tool to solve the missing cases in Theorem 2.2.5 and many of the results we show in this thesis could be solved, in an easier way, if this were true.

Remark 2.3.3. Assuming a positive answer to Problem 2.3.1 it is not difficult to complete all the remaining cases in Theorem 2.2.5 (i.e., for $p \in [2, m]$ and $q \in [2, \infty]$). In that case we have,

$$\begin{cases} A^m_{p,q}(n) \sim n^{\frac{1}{q}} & \text{for } (\overline{F}) : [\frac{1}{m} \leqslant \frac{1}{p} \leqslant \frac{1}{2} \ \wedge \ \frac{1}{q} \leqslant \frac{m}{2-m} \cdot \frac{1}{p} + \frac{m}{2m-4}], \\ A^m_{p,q}(n) \ll n^{m(\frac{1}{p}+\frac{1}{q}-\frac{1}{2})-\frac{1}{q}} & \text{for } (\overline{G}) : [\frac{1}{m} \leqslant \frac{1}{p} \leqslant \frac{1}{2} \ \wedge \ \frac{m}{2-m} \cdot \frac{1}{p} + \frac{m}{2m-4} \leqslant \frac{1}{q} \leqslant \frac{1}{2}], \end{cases}$$

Moreover, the power of n in $(\overline{G})$ cannot be improved.

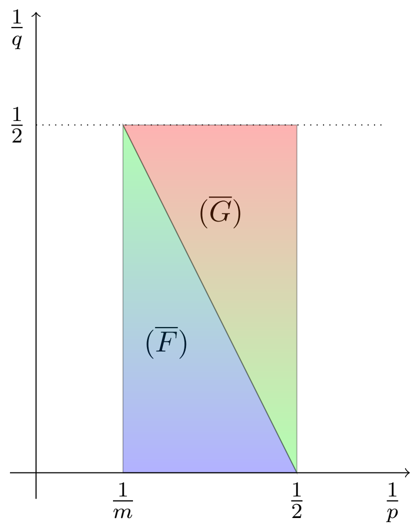

Figure 2.2: Graphical overview of the cases treated in Remark 2.3.3.

The previous remark is one very concrete motivation to investigate around the Problem 2.3.1. We don't give a proof of Remark 2.3.3 because we know, as we will now show, this problem has a negative answer. Nevertheless we will show the lower bounds coincide with Remark 2.3.3. For the region $(\overline{G})$ this assertion can be proved using Steiner polynomials: take a polynomial P as in Theorem 2.2.3, then

$$n^{\frac{m-1}{r}} \sim |P|_r \leqslant A^m_{p,r}(n)\|P\|_{\mathcal{P}(^m\ell_p^n)} \ll A^m_{p,r}(n)\log(n)^{\frac{3}{p}} n^{m(\frac{1}{2}-\frac{1}{p})} \ll A^m_{p,r}(n) n^{m(\frac{1}{2}-\frac{1}{p})+\varepsilon},$$

for every $\varepsilon > 0$, then we have

$$n^{m(\frac{1}{r}+\frac{1}{p}-\frac{1}{2})-\frac{1}{r}-\varepsilon} \ll A^m_{p,r}(n).$$

For the region $(\overline{F})$ the same argument with the same polynomial used in Theorem 2.2.5 in region (F) work.

Corollary 2.3.4. *The answer to the question in Problem 2.3.1 is negative in general. In particular, for $m \geqslant 3$, $q > m$, $(m-1)q' \leqslant p_1 < m$, $mq' \leqslant p_0$ and θ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \leqslant m$, there is no bounded linear isomorphism*

$$T : \mathcal{P}(^m[\ell_{p_0}, \ell_{p_1}]_\theta) \to [\mathcal{P}(^m\ell_{p_0}), \mathcal{P}(^m\ell_{p_1})]_\theta,$$

such that it induces linear isomorphism $T_n : \mathcal{P}(^m[\ell^n_{p_0}, \ell^n_{p_1}]_\theta) \to [\mathcal{P}(^m\ell^n_{p_0}), \mathcal{P}(^m\ell^n_{p_1})]_\theta$.

Proof. Notice for the induced family of linear isomorphism we have $\|T_n\| \leqslant \|T\|$ and $\|T_n^{-1}\| \leqslant \|T^{-1}\|$ for every $n \in \mathbb{N}$.

Observe in Theorem 2.2.5 that $(\frac{1}{p_0}, \frac{1}{q}) \in (A)$ and $(\frac{1}{p_1}, \frac{1}{q}) \in (F)$ with $p_1 > m$, then

$$\|id : \mathcal{P}(^m\ell^n_{p_0}) \to (\mathcal{P}(^m\mathbb{C}^n), |\cdot|_q)\| = A^m_{p_0,q}(n) \leqslant C_{m,p_0,q},$$
$$\|id : \mathcal{P}(^m\ell^n_{p_1}) \to (\mathcal{P}(^m\mathbb{C}^n), |\cdot|_q)\| = A^m_{p_1,q}(n) \leqslant C_{m,p_1,q} n^{\frac{1}{q}}.$$

Using the multilinear interpolation Theorem 1.4.1 we have

$$\|id : [\mathcal{P}(^m\ell^n_{p_0}), \mathcal{P}(^m\ell^n_{p_1})]_\theta \to [(\mathcal{P}(^m\mathbb{C}^n), |\cdot|_q), (\mathcal{P}(^m\mathbb{C}^n), |\cdot|_q)]_\theta\| \leqslant C^\theta_{m,p_0,q} C^{1-\theta}_{m,p_1,q} n^{\frac{1-\theta}{q}},$$

for every $0 < \theta < 1$.

Assuming a positive answer to the question posed in Problem 2.3.1 and using Theorem 1.4.1 we would have that for, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\theta \in (0,1)$ and $P \in \mathcal{P}(^m\mathbb{C}^n)$,

$$\begin{aligned} |P|_q &\ll n^{\frac{1-\theta}{q}} \|P\|_{[\mathcal{P}(^m\ell^n_{p_0}), \mathcal{P}(^m\ell^n_{p_1})]_\theta} \\ &\leqslant n^{\frac{1-\theta}{q}} \|T\| \|P\|_{\mathcal{P}(^m\ell^n_p)}, \end{aligned}$$

then

$$A^m_{p,q}(n) \ll n^{\frac{1-\theta}{q}}.$$

Choosing θ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \leqslant m$ we have $(p,q) \in (F)$, this contradicts the lower bound from region (F) in Theorem 2.2.5 . □

Notice that Corollary 2.3.4 denies the existence of an isomorphism between the spaces $\mathcal{P}({}^m[\ell_{p_0}, \ell_{p_1}]_\theta)$ and $[\mathcal{P}({}^m\ell_{p_0}), \mathcal{P}({}^m\ell_{p_1})]_\theta$ with some conditions on p_0 and p_1. This does not include the values of p_0 and p_1 that would disable the use of complex interpolation needed to prove the result given in Remark 2.3.3. In [BM19, Theorem 3.10] the authors prove that for those values there is no such isomorphism either. In particular they prove there a theorem for a bigger family of interpolation methods, that for the complex methods can be expressed in the following way.

Theorem 2.3.5. *For $m \geqslant 2$ and $2 \leqslant h \leqslant m$ given $1 \leqslant p_1 < h < p_0$ and $0 < \theta < 1$ there is no Banach spaces isomorphism between $\mathcal{P}({}^m[\ell_{p_0}^n, \ell_{p_1}^n]_\theta)$ and $[\mathcal{P}({}^m\ell_{p_0}^n), \mathcal{P}(\ell_{p_1}^n)]_\theta)$.*

In [BPR18] the authors also gave some answers to similar questions concerning complex interpolation between the tensor products of Banach spaces.

2.3.2 The multivariable von Neumann's inequality

A classical inequality in operator theory, due to von Neumann [vN51], asserts that if T is a linear contraction on a complex Hilbert space $\mathcal{H}$ (i.e., its operator norm is less than or equal to one) then

$$\|P(T)\|_{\mathcal{L}(\mathcal{H})} \leqslant \sup\{|P(z)| : z \in \mathbb{C}, |z| \leqslant 1\},$$

for every polynomial P in one (complex) variable.

Using dilation theory (see [SN74]), Ando [And63] exhibited an analogue inequality for polynomials in two commuting contractions. However Varopoulos [Var74] showed that von Neumann's inequality cannot be extended for three or more commuting contractions.

It is an open problem of great interest in operator theory (see for example [Ble01, Pis01]) to determine whether there exists a constant $K(n)$ that adjusts von Neumann's inequality. More precisely, it is unknown whether or not there exists a constant $K(n)$ such that

$$\|P(T_1, \ldots, T_n)\|_{\mathcal{L}(\mathcal{H})} \leqslant K(n) \ \sup\{|P(z_1, \ldots, z_n)| : |z_i| \leqslant 1\}, \tag{2.27}$$

for every polynomial P in n variables and every n-tuple $(T_1, \ldots, T_n)$ of commuting contractions in $\mathcal{L}(\mathcal{H})$.

Dixon [Dix76] studied the multivariable von Neumann's inequality restricted to homogeneous polynomials and, together with Mantero [MT79], studied some variations of this problem. One of them is to determine the asymptotic behavior of the best possible constant $c(n) = c_{m,p,q}(n)$ such that

$$\|P(T_1, \ldots, T_n)\|_{\mathcal{L}(\mathcal{H})} \leqslant c(n)\|P\|_{\mathcal{P}({}^m\ell_q^n)},$$

for every n-tuple $T_1, \ldots, T_n$ of commuting operators on a Hilbert space satisfying

$$\sum_{i=1}^{n} \|T_i\|_{\mathcal{L}(\mathcal{H})}^p \leqslant 1, \tag{2.28}$$

and any m-homogeneous polynomial on n variables, P. Some lower bounds were proven there and also some upper bounds were given for the case $p = q$. We will use Theorem 2.2.5 and Theorem 2.2.6 to show upper bounds for $c(n)$ for any $1 \leqslant p, q \leqslant \infty$.

Recall that given a bilinear form $a : X_1 \times X_2 \to \mathbb{C}$ its uniform norm is

$$\|a\|_{Bil(X_1\times X_2)} := \sup_{(x_1,x_2)\in B_{X_1}\times B_{X_2}} |a(x_1,x_2)|.$$

We need the following lemma from [MT79] which is an easy consequence of the Grothendieck inequality. We prove it for the sake of completeness.

Lemma 2.3.6. *For $i = 1,\dots,N$, $j = 1,\dots,M$ let x_i, y_j be vectors in some Hilbert space $\mathcal{H}$ such the $\sum_{i=1}^N \|x_i\|_{\mathcal{H}}^p \leqslant 1$ and $\sum_{j=1}^M \|y_j\|_{\mathcal{H}}^p \leqslant 1$, and let $(a_{i,j})_{i,j} \in \mathbb{C}^{N\times M}$. Then*

$$\left|\sum_{i,j} a_{ij}\langle x_i, y_j\rangle\right| \leqslant K_G \|a\|_{Bil(\ell_p^N\times \ell_p^M)},$$

where K_G denotes the Grothendieck constant and a is the bilinear form on $\mathbb{C}^N\times\mathbb{C}^M$ whose coefficients are the a_{ij}'s.

Proof.

$$\begin{aligned}\left|\sum_{i,j} a_{i,j}\langle x_i, y_j\rangle\right| &\leqslant \left|\sum_{i,j} a_{i,j}\|x_i\|_{\mathcal{H}}\|y_j\|_{\mathcal{H}}\langle \frac{x_i}{\|x_i\|_{\mathcal{H}}}, \frac{y_j}{\|y_j\|_{\mathcal{H}}}\rangle\right| \\ &\leqslant K_G \sup\{\sum_{i,j} a_{i,j}\|x_i\|_{\mathcal{H}}\|y_j\|_{\mathcal{H}}\beta_i\gamma_j \ : \ \beta\in B_{\ell_\infty^N}, \gamma\in B_{\ell_\infty^M}\} \\ &\leqslant K_G\|a\|_{Bil(\ell_p^N\times\ell_p^M)}.\end{aligned}$$

□

Proposition 2.3.7. *Let $T_1,\dots,T_n$ be commuting operators on a Hilbert space $\mathcal{H}$ satisfying (2.28) and $P\in\mathcal{P}(^m\mathbb{C}^n)$. Then*

$$\|P(T_1,\dots,T_n)\|_{\mathcal{L}(\mathcal{H})} \leqslant C A_{q,p'}^{m-1}(n)\|P\|_{\mathcal{P}(^m\ell_q^n)},$$

where $C > 0$ is constant independent of n.

Proof. Let $a_{\mathbf{i}}$, $\mathbf{i}\in\mathcal{M}(m,n)$ be the coefficients of the symmetric m-linear form a associated to P, and let x, y be unit vectors in $\mathcal{H}$. Note that we may also view a as a bilinear form on $\mathbb{C}^{n^{m-1}}\times\mathbb{C}^n$, then by the previous lemma,

$$\begin{aligned}\left|\sum_{\mathbf{i}\in\mathcal{M}(m,n)} a_{\mathbf{i}}\langle T_{i_1}\dots T_{i_m}x, y\rangle\right| &= \left|\sum_{(\mathbf{i},j)\in\mathcal{M}(m-1,n)\times\{1,\dots,n\}} a_{(\mathbf{i},j)}\langle T_{i_1}\dots T_{i_{m-1}}x, T_j^* y\rangle\right| \\ &\leqslant K_G\|a\|_{Bil(\ell_p^{n^{m-1}}\times\ell_p^n)} = K_G \sup_{\beta\in B_{\ell_p^n}}\left(\sum_{\mathbf{i}\in\mathcal{M}(m-1,n)}\left|\sum_{j=1}^n a_{(\mathbf{i},j)}\beta_j\right|^{p'}\right)^{1/p'}.\end{aligned}$$

Observe that $\sum_{j=1}^n a_{(\mathbf{i},j)}\beta_j$ are the coefficients of the $(m-1)$-linear form a_β which is obtained by fixing one variable of a at β, that is, $a_\beta(v_1,\dots,v_{m-1}) = a(\beta, v_1,\dots,v_{m-1})$. Also, the

p'-norm of the coefficients of a_β is less than or equal to the p'-norm of the coefficients of the associated polynomial P_β. Thanks to Theorem 1.2.7 it follows $\|P_\beta\|_{\mathcal{P}(^{m-1}\ell_q^n)} \leqslant e\|P\|_{\mathcal{P}(^m\ell_q^n)}$, then taking supremum over $x, y \in B_{\mathcal{H}}$ we have

$$\begin{aligned}\|P(T_1, \ldots, T_n)\|_{\mathcal{L}(\mathcal{H})} &\leqslant K_G \sup_{\beta \in B_{\ell_p^n}} A_{q,p'}^{m-1}(n)\|P_\beta\|_{\mathcal{P}(^{m-1}\ell_q^n)} \\ &\leqslant K_G e A_{q,p'}^{m-1}(n)\|P\|_{\mathcal{P}(^m\ell_q^n)}. \qquad \square\end{aligned}$$

Remark 2.3.8. Taking $p = q$ and using Theorem 2.2.5 we recover the inequality proved in [MT79], that is, $c(n) \ll n^{\frac{m-2}{p'}}$ if $p \leqslant 2$ and $c(n) \ll n^{\frac{m-2}{2}}$ if $p \geqslant 2$.

We also have the following corollary.

Corollary 2.3.9. *Let $T_1, \ldots, T_n$ be commuting operators on a Hilbert space $\mathcal{H}$ satisfying (2.28). If $[\frac{1}{2} \leqslant \frac{1}{p'} \leqslant \frac{m}{2(m-1)} - \frac{1}{q}]$ or $[\frac{1}{p'} \leqslant \frac{1}{2} \ \wedge \ \frac{m-1}{q} \leqslant 1 - \frac{1}{p'}]$, we have that*

$$\|P(T_1, \ldots, T_n)\|_{\mathcal{L}(\mathcal{H})} \leqslant D\|P\|_{\mathcal{P}(^m\ell_q^n)},$$

for every m-homogeneous polynomial P, where D is constant independent of n.

Another variant studied in [MT79] is to determine the best possible constant $d(n) = d_{m,p,q}(n)$ such that

$$\|P(T_1, \ldots, T_n)\|_{\mathcal{L}(\mathcal{H})} \leqslant d(n)\|P\|_{\mathcal{P}(^m\ell_q^n)},$$

for every m-homogeneous polynomial in n variables, P, and every n-tuple $T_1, \ldots, T_n$ of commuting operators on a Hilbert space $\mathcal{H}$ satisfying

$$\left(\sum_{i=1}^n |\langle T_i x, y\rangle|^p\right)^{1/p} \leqslant \|x\|_{\mathcal{H}}\|y\|_{\mathcal{H}}, \tag{2.29}$$

for any vectors $x, y \in \mathcal{H}$. Note that (2.29) is equivalent to $\|\sum_{i=1}^n T_i\beta_i\|_{\mathcal{L}(\mathcal{H})} \leqslant \|\beta\|_{p'}$, for every $\beta \in \mathbb{C}^n$.

Lemma 2.3.10. *Let $T_1, \ldots, T_n \in \mathcal{L}(\mathcal{H})$ be operators satisfying (2.29), and let $x, y \in \mathcal{H}$. Then if Q is the m-homogeneous polynomial in n variables defined by*

$$Q(z) = \sum_{\mathbf{i} \in \mathcal{M}(m,n)} \langle T_{i_1} \ldots T_{i_m} x, y\rangle z_{i_1} \ldots z_{i_m},$$

we have $\|Q\|_{\mathcal{P}(^m\ell_{p'}^n)} \leqslant \|x\|_{\mathcal{H}}\|y\|_{\mathcal{H}}$.

Proof. By a simple calculation we have

$$\begin{aligned}
\|Q\|_{\mathcal{P}(^m\ell_{p'}^n)} &= \sup_{z\in B_{\ell_{p'}^n}} \left| \sum_{\mathbf{i}\in\mathcal{M}(m,n)} \langle T_{i_1}\dots T_{i_m}x,y\rangle z_{i_1}\dots z_{i_m} \right| \\
&= \sup_{z\in B_{\ell_{p'}^n}} \left| \left\langle \sum_{\mathbf{i}\in\mathcal{M}(m,n)} T_{i_1}\dots T_{i_m} z_{i_1}\dots z_{i_m} x, y \right\rangle \right| \\
&\leqslant \sup_{z\in B_{\ell_{p'}^n}} \left| \left\langle \left(\sum_{l=1}^n z_l T_l\right)^m x, y \right\rangle \right| \leqslant \sup_{z\in B_{\ell_{p'}^n}} \left\| \sum_{l=1}^n z_l T_l \right\|^m \|x\|_{\mathcal{H}}\|y\|_{\mathcal{H}} \leqslant \|x\|_{\mathcal{H}}\|y\|_{\mathcal{H}}. \ \square
\end{aligned}$$

Proposition 2.3.11. *Let $T_1,\dots,T_n$ be commuting operators on a Hilbert space $\mathcal{H}$ satisfying (2.29) and $P\in\mathcal{P}(^m\mathbb{C}^n)$. Then*

$$\|P(T_1,\dots,T_n)\|_{\mathcal{L}(\mathcal{H})} \leqslant A_{q,r}^m(n)A_{p',r'}^m(n)\|P\|_{\mathcal{P}(^m\ell_q^n)}.$$

Proof. Let $a_{\mathbf{i}}$ be the coefficients of the symmetric m-linear form a associated to P and x,y unit vectors in $\mathcal{H}$. Then by the previous lemma and the fact that the r-norm of the coefficients of a is less than or equal to the r-norm of the coefficients of the associated polynomial P, we have

$$\begin{aligned}
\left| \sum_{\mathbf{i}\in\mathcal{M}(m,n)} a_{\mathbf{i}}\langle T_{i_1}\dots T_{i_m}x,y\rangle \right| &\leqslant \left(\sum_{\mathbf{i}\in\mathcal{M}(m,n)} |a_{\mathbf{i}}|^r \right)^{1/r} \left(\sum_{\mathbf{i}\in\mathcal{M}(m,n)} |\langle T_{i_1}\dots T_{i_m}x,y\rangle|^{r'} \right)^{1/r'} \\
&\leqslant A_{q,r}^m(n)\|P\|_{\mathcal{P}(^m\ell_q^n)} A_{p',r'}^m(n). \qquad \square
\end{aligned}$$

Remark 2.3.12. Taking $p=q=r'$ and using Theorem 2.2.5 we recover the inequality proved in [MT79, Proposition 20], that is $d(n) \ll n^{(m-1)(\frac{1}{p'}+\frac{1}{2})}$ if $p\leqslant 2$ and $d(n) \ll n^{(m-1)(\frac{1}{p}+\frac{1}{2})}$ if $p\geqslant 2$. Note also that, in the last proposition, we have bounds that do not depend on n for some combinations of p and q, e.g. for $(p,q)=(1,\infty)$.

Chapter 3

Monomial convergence

A classical result in complex analysis in one variable, and arguably the most important of the theory, states that every holomorphic function can be represented locally as a power series. More precisely, for $f : U \subset \mathbb{C} \to \mathbb{C}$ holomorphic on U a complex open set and $z_0 \in U$ there is $r = r(z_0) > 0$ and a sequence $(c_m)_{m \geqslant 1} \subset \mathbb{C}$ such that

$$f(z) = \sum_{m \geqslant 1} c_m (z - z_0)^m + f(z_0),$$

for every $z \in B_r(z_0)$. This fact is also true for holomorphic functions on open sets on $\mathbb{C}^n$ as it can be seen in Theorem 1.3.3.

There is a generalization of this fact to infinite dimentions. Given an open set U in a Banach space X and a holomorphic function $f : U \subset X \to \mathbb{C}$, for every $z_0 \in U$ there are m-homogeneous polynomials $P_m = P_m(f, z_0) \in \mathcal{P}(^m X)$ such that

$$f(z) = \sum_{m \geqslant 1} P_m(z - z_0) + f(z_0), \tag{3.1}$$

for every z in a neighborhood of z_0.

This is the only description we may expect in the Taylor series sense for a holomorphic function on an open set in general Banach space, but some structure on the Banach space might allow us to have another insight. The structure of Banach sequence spaces is exactly what we need to define the monomials.

3.1 Definitions and first results

Given $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ and a Banach sequence space X we may think of the monomial defined by α as the mapping

$$\begin{aligned} (\cdot)^\alpha : X &\to \mathbb{C} \\ z &\mapsto z^\alpha. \end{aligned}$$

This mapping is an m-homogeneous polynomial on X but not every m-homogeneous polynomial is a monomial nor a finite linear combination of them. For example consider the linear functional

$$\begin{aligned} \ell_1 &\to \mathbb{C} \\ z &\to \sum_{k \geqslant 1} z_k, \end{aligned}$$

which is clearly not a finite linear combination of monomials but, nevertheless, it is a limit of linear combination of the them. It is natural to ask whether this is a general fact. We will now give rigorous sense of this question and generalize it to holomorphic functions.

Let f be a holomorphic function on some Reinhardt domain $\mathcal{R}$ in a Banach sequence space X. Fixed $n \in \mathbb{N}$, the restriction of f to $\mathcal{R}_n = \mathcal{R} \cap \mathbb{C}^n$ (which is a Reinhardt domain) is holomorphic and, therefore by Theorem 1.3.3 has a *monomial series expansion* with coefficients $(a_\alpha^{(n)}(f))_{\alpha \in \mathbb{N}_0^n}$, i.e.,

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha^{(n)}(f) z^\alpha,$$

for every $z \in \mathcal{R}_n$. Using that $\mathcal{R}_n \subset \mathcal{R}_{n+1}$ an the uniqueness of the power series coefficients on finite many variables it is easy to see $a_\alpha^{(n)}(f) = a_\alpha^{(n+1)}(f)$ for $\alpha \in \mathbb{N}_0^n \subset \mathbb{N}_0^{n+1}$. In other words, we have a unique sequence $(a_\alpha(f))_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}$, such that

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z^\alpha \tag{3.2}$$

for all $n \in \mathbb{N}$ and all $z \in \mathcal{R}_n$. This power series is called the monomial series expansion of f. Sometimes it will be convenient to describe this monomial expansion in terms of the alternative writing of the monomials $\{z_{\mathbf{j}} : \mathbf{j} \in \mathcal{J}\}$ where $\mathcal{J} = \cup_{m \in \mathbb{N}_0} \mathcal{J}(m)$. In this cases we will denote $c_{\mathbf{j}}(f) = a_\alpha(f)$ when $\mathbf{j} = F(\alpha)$ (see (1.6)).

Problem 3.1.1. *Given an holomorphic function f in a Reinhardt domain $\mathcal{R}$ on a Banach sequence space X, a number of questions arise.*

(1) Is it true that $f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z^\alpha$ for every $z \in \mathcal{R}$?

(2) If not, given a family of holomorphic functions on $\mathcal{R}$, can we describe the sets

$$\left\{ z \in \mathcal{R} : f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z^\alpha \text{ for every } f \text{ in the family} \right\} ?$$

First it is necessary to give a precise meaning to the expression $f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z^\alpha$ whenever z has not finite support. There is no natural order in the set $\mathbb{N}_0^{(\mathbb{N})}$, also we would like the sum not to depend on some rather artificial order. We say that $\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f)z^\alpha$ converges to some element if the convergence is unconditional, or equivalently, if the convergence is absolute.

One could expect that in the settings where the approaches given by equation (3.1) and equation (3.2) coexist they are equivalent, just as in the finite dimensional setting. But this is not the case. When dealing with a totally different problem, related to the convergence of Dirichlet series, Toeplitz gave in [Toe13] an example that, to what we are concerned here, provided a holomorphic function on c_0 and a point in c_0 for which the monomial expansion does not converge absolutely. This shows that there are holomorphic functions for which its monomial description is bad (the converse, however, holds true: every function that may be described by its monomial series is holomorphic).

Within the theory of complex analysis in one variable the absolute convergence of the power series plays an important role determining the radius of convergence, here it will be crucial too. Also the unconditional nature of the convergence implies that $\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f)z^\alpha \in \mathbb{C}$ is convergent if and only if it converges absolutely, as this two concepts are equivalent in $\mathbb{C}$. With this in mind the following question arises naturally: for which z's does the monomial expansion of every holomorphic function in a given family converges absolutely? From equation (3.2) we know that this happens for every $z \in \mathcal{R}_n$ but, can we describe the maximal set for this to happen? Ryan showed in [Rya87] that the monomial expansion of every holomorphic function on ℓ_1 converges at every $z \in \ell_1$. Later, Lempert in [Lem99] proved that the monomial expansion of every holomorphic function on ρB_{ℓ_1} (for $\rho > 0$) converges at every $z \in \rho B_{\ell_1}$. This is somehow an extreme case, where the analytic and differential approaches coincide. What happens in other spaces or if we consider smaller families of holomorphic functions? To deal with this questions it was defined in [DMP09] the set of monomial convergence for a given holomorphic function. Let $\mathcal{R}$ be a Reinhardt domain on some Banach sequence space X and f a holomorphic function on $\mathcal{R}$, the *set of monomial convergence* for f is

$$mon f := \Big\{ z \in \mathbb{C}^{\mathbb{N}} : \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |a_\alpha(f)z^\alpha| < \infty \Big\},$$

i.e., those elements of $\mathcal{R}$ for which the power series of f converge absolutely. For a *family of holomorphic functions* on $\mathcal{R}$ named $\mathcal{F}(\mathcal{R})$ its set of monomial convergence is

$$mon\mathcal{F}(\mathcal{R}) := \Big\{ z \in \mathbb{C}^{\mathbb{N}} : \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} \left|a_\alpha(f)z^\alpha\right| < \infty \ \text{ for every } f \in \mathcal{F}(\mathcal{R}) \Big\}.$$

There are many families of holomorphic functions for which is interesting to find its set of monomial convergence. Given a sequence Banach space X the following families are the most natural to study in this sense:

- $\mathcal{P}(^m X)$, m-homogeneous polynomials on X.
- $\mathcal{A}_u(B_X)$, the Banach algebra of all uniformly continuous holomorphic functions on the unit ball of X.
- $H_\infty(B_X)$, the Banach space of holomorphic functions on the ball of X which are also bounded.
- $H_b(X)$, the Fréchet space of the entire functions on X which are also bounded on the bounded set of X.

Let $f : \mathcal{R} \subset X \to \mathbb{C}$ be a holomorphic function on a Reinhardt domain $\mathcal{R}$ of a Banach sequence space X. It makes sense to compute $\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |a_\alpha(f) z^\alpha|$ for every $z \in \mathbb{C}^{\mathbb{N}}$, even if f is not defined outside $\mathcal{R}$ and $z \notin \mathcal{R}$. On the other hand the condition $f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z^\alpha$ have meaning only for $z \in \mathcal{R}$. Then in general, given $\mathcal{F}(\mathcal{R})$ a family of holomorphic functions, $mon\mathcal{F}(\mathcal{R})$ is not exactly the set

$$\Big\{ z \in \mathcal{R} : f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z^\alpha \ \text{ for every } f \in \mathcal{F}(\mathcal{R}) \Big\}.$$

Anyway for many natural Banach sequence spaces and families of holomorphic functions on Reinhardt domains in those spaces these two sets coincide. In particular this is true for those families in which we focus on this thesis.

Proposition 3.1.2. *Given $1 < p, q \leqslant \infty$ for $X = \ell_{p,q}$ we have*

$$mon\mathcal{F}(\mathcal{R}) = \Big\{ z \in \mathcal{R} : f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z^\alpha \ \textit{ for every } f \in \mathcal{F}(\mathcal{R}) \Big\},$$

for $\mathcal{F}(\mathcal{R})$ being $H_b(X), H_\infty(B_X)$ or $\mathcal{P}(^m X)$ for any $m \in \mathbb{N}$.

We give a proof of Proposition 3.1.2 on Appendix A.

We will now give a series of basic results for the set of monomial convergence. A very useful but simple fact is that given two families of holomorphic functions $\mathcal{F}_1(\mathcal{R}) \subset \mathcal{F}_2(\mathcal{R})$ on certain Reinhardt domain $\mathcal{R}$ we have

$$mon\mathcal{F}_2(\mathcal{R}) \subset mon\mathcal{F}_1(\mathcal{R}). \tag{3.3}$$

For a bounded Reinhardt domain $\mathcal{R}$ in a Banach sequence space X, $H_\infty(\mathcal{R})$ is the family of the holomorphic functions on $\mathcal{R}$ which are bounded on $\mathcal{R}$. Recall that $H_\infty(\mathcal{R})$ is a Banach space with the norm given by $\|f\|_{\mathcal{R}} = \sup_{z \in \mathcal{R}} |f(z)|$ for every $f \in H_\infty(\mathcal{R})$. Given a subfamily $\mathcal{F}(\mathcal{R}) \subset H_\infty(\mathcal{R})$ we say $\mathcal{F}(\mathcal{R})$ is closed in $H_\infty(\mathcal{R})$ when it is a closed subspace with that norm.

Remark 3.1.3. Given a bounded Reinhardt domain $\mathcal{R}$, a closed family $\mathcal{F}(\mathcal{R})$ in $H_\infty(\mathcal{R})$ and $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$, the linear mapping

$$\mathcal{F}(\mathcal{R}) \to \mathbb{C} \tag{3.4}$$

$$f \mapsto a_\alpha(f), \tag{3.5}$$

is bounded.

Proof. Observe that for $f \in H_\infty(\mathcal{R})$ and fixed $\alpha = (\alpha_1, \ldots, \alpha_n, 0, \ldots) \in \mathbb{N}_0^{(\mathbb{N})}$, thanks to Theorem 1.3.3, we have

$$a_\alpha(f) = \frac{1}{(2\pi i)^n} \int_{|\xi_1|=\rho_1} \cdots \int_{|\xi_n|=\rho_n} \frac{f(\xi_1, \ldots, \xi_n)}{(\xi_1 - z_1)^{\alpha_1+1} \cdots (\xi_n - z_n)^{\alpha_n+1}} d\xi,$$

with $\rho_1\mathbb{D} \times \cdots \times \rho_n\mathbb{D} \subset \mathcal{R}$. We have then

$$|a_\alpha(f)| \leqslant \frac{1}{(2\pi)^n \prod_{k=1}^n \rho_k^{\alpha_k}} \|f\|_{\rho_1\mathbb{D}\times\cdots\times\rho_n\mathbb{D}} \leqslant C(\alpha)\|f\|_{\mathcal{R}}.$$ □

For a closed subfamily $\mathcal{F}(\mathcal{R})$ in $H_\infty(\mathcal{R})$ the following equivalence is a powerful tool that translates the fact of an element being on $mon\mathcal{F}(\mathcal{R})$ to an inequality over the whole family.

Proposition 3.1.4. *Given a bounded Reinhardt domain $\mathcal{R}$ in a Banach sequence space X and $\mathcal{F}(\mathcal{R})$ a closed subfamily of $H_\infty(\mathcal{R})$ the following are equivalent:*

- $z \in mon\mathcal{F}(\mathcal{R})$.
- *There is some constant $C_z > 0$ such that*

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |a_\alpha(f) z^\alpha| \leqslant C_z \|f\|_{\mathcal{R}}, \tag{3.6}$$

  for every $f \in \mathcal{F}(\mathcal{R})$.

Proof. Given $z \in \mathcal{R}$ meeting (3.6) it clearly holds $\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |a_\alpha(f) z^\alpha| < \infty$ for every $f \in \mathcal{F}(\mathcal{R})$. For the other implication consider the linear mapping

$$\Phi_z : \mathcal{F}(\mathcal{R}) \to \ell_1\left(\mathbb{N}_0^{(\mathbb{N})}\right)$$

$$f \mapsto (a_\alpha(f) z^\alpha)_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}},$$

that is well defined as $z \in mon\mathcal{F}(\mathcal{R})$. For $f_n \in \mathcal{F}(\mathcal{R})$ such that $f_n \to f \in \mathcal{F}(\mathcal{R})$ and $\Phi_z(f_n) \to b = (b_\alpha)_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}$ we have, for any $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$, that $a_\alpha(f_n) \to b_\alpha$. By Remark 3.1.3 it holds $a_\alpha(f_n) \to a_\alpha(f)$, and due to the uniqueness of the limit $b = \Phi_z(f)$ the graph of Φ_z is closed. As $\mathcal{F}(\mathcal{R})$ is a closed subfamily of $H_\infty(\mathcal{R})$ and $H_\infty(\mathcal{R})$ is a Banach space, then $\mathcal{F}(\mathcal{R})$ is also Banach. Using the closed graph theorem for Banach spaces it follows Φ_z is bounded, which is exactly what we wanted. □

For every $m \in \mathbb{N}$ the space $\mathcal{P}(^m X)$ is a closed subfamily of $H_\infty(B_X)$ for every Banach sequence space X. This is also the case for $\mathcal{A}_u(B_X)$. On the other hand, although as sets we have $H_b(X) \subset H_\infty(B_X)$, $H_b(X)$ is not a closed subspace of $H_\infty(B_X)$. In Chapter 6 we will need and provide a new version of Proposition 3.1.4 for the case of $H_b(X)$.

Now we present a point of view that will play an important role describing sets of monomial convergence of certain families of holomorphic functions in terms of subfamilies of $H_\infty(B_{\ell_\infty})$. Let $\mathcal{R}$ be a bounded Reinhardt domain and $\mathcal{F}(\mathcal{R})$ a closed subfamily of $H_\infty(\mathcal{R})$. Given $f \in \mathcal{F}(\mathcal{R})$ and $w \in \mathcal{R}$ we define $f_w \in H_\infty(B_{\ell_\infty})$ as $f_w(z) = f(w \cdot z)$.

Remark 3.1.5. For every Reinhardt domain $\mathcal{R}$ and every $w \in \mathcal{R}$ it holds

$$\|f_w\|_{B_{\ell_\infty}} \leqslant \|f\|_{\mathcal{R}},$$

and $a_\alpha(f_w) = a_\alpha(f) w^\alpha$.

Proof. Notice that given $w \in \mathcal{R}$ and $z \in B_{\ell_\infty}$ it holds $w \cdot z \in \mathcal{R}$ as $|(w \cdot z)_k| = |w_k||z_k| \leqslant |w_k|$ for every $k \in \mathbb{N}$. Then by definition it follows

$$\|f_w\|_{B_{\ell_\infty}} = \sup_{z \in B_{\ell_\infty}} |f_w(z)| = \sup_{z \in B_{\ell_\infty}} |f(w \cdot z)| \leqslant \sup_{u \in \mathcal{R}} |f(u)| = \|f\|_{\mathcal{R}}.$$

Fix $n \in \mathbb{N}$, for $z \in \mathcal{R}_n$ we have $w \cdot z \in \mathcal{R}_n$ and then

$$f_w(z) = f(w \cdot z) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha(f)(w \cdot z)^\alpha = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha(f) w^\alpha z^\alpha.$$

By the uniqueness of the monomial expansion in finite complex variables we have

$$a_\alpha(f_w) = a_\alpha(f) w^\alpha,$$

for every $\alpha \in \mathbb{N}_0^n$ for every $n \in \mathbb{N}$, then it holds for every $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$. ☐

Given a Reinhardt domain $\mathcal{R}$ on a Banach sequence space X and fixed an element $w \in \mathcal{R}$ we have

- For $\mathcal{R} = X$ and $f \in H_b(X)$ then $f_w \in H_b(\ell_\infty)$.
- For $\mathcal{R}$ bounded and $f \in H_\infty(\mathcal{R})$ then $f_w \in H_\infty(B_{\ell_\infty})$.
- For $\mathcal{R} = X$ and $P \in \mathcal{P}(^m X)$ then $P_w \in \mathcal{P}(^m \ell_\infty)$.

Given a family of holomorphic functions $\mathcal{F}(\mathcal{R})$ on a Reinhardt domain $\mathcal{R}$ we define

$$[\mathcal{F}(\mathcal{R})]_\infty := \{f_w : \text{ for every } w \in \mathcal{R} \text{ and every } f \in \mathcal{F}(\mathcal{R})\}.$$

Notice then that it holds the set inclusions $[H_b(X)]_\infty \subset H_b(\ell_\infty)$, $[H_\infty(\mathcal{R})]_\infty \subset H_\infty(B_{\ell_\infty})$ and $[\mathcal{P}(^m X)]_\infty \subset \mathcal{P}(^m \ell_\infty)$.

Lemma 3.1.6. *For every Reinhardt domain $\mathcal{R}$ and every family of holomorphic functions $\mathcal{F}(\mathcal{R})$ it holds*

$$\mathcal{R} \cdot mon[\mathcal{F}(\mathcal{R})]_\infty \subset mon\mathcal{F}(\mathcal{R}).$$

Proof. Given $w \in \mathcal{R}$ and $f \in \mathcal{F}(\mathcal{R})$ it holds $f_w \in [\mathcal{F}(\mathcal{R})]_\infty$. Then for $z \in mon[\mathcal{F}(\mathcal{R})]_\infty$, as $a_\alpha(f_w) = a_\alpha(f)w^\alpha$ we have

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |a_\alpha(f)(wz)^\alpha| = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |a_\alpha(f)w^\alpha z^\alpha| = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |a_\alpha(f_w)z^\alpha| < \infty. \qquad \square$$

3.2 Some characterizations

The only natural family of holomorphic functions for which the set of monomial convergence is known for every Banach sequence space is the one given by its dual space, as presented in equation (A.0.3). In general it is difficult to have a finished description of these sets in terms of some known space or set. As we highlighted before the efforts of Ryan [Rya87] and Lempert [Lem99] toghether made the first result which characterized the set of monomial convergence of a family of holomorphic function on an infinite dimensional Banach sequence space (other than the dual space of any Banach sequence space). The following theorem is a corollary of those investigations.

Theorem 3.2.1. *For every $\rho > 0$ it holds $monH_\infty(\rho B_{\ell_1}) = \rho \cdot B_{\ell_1}$. Also for every $m \in \mathbb{N}$ $\mathcal{P}(^m\ell_1) = \ell_1$.*

For the other end of the range for Lorentz sequence spaces in [BDF$^+$17] the authors proved the following two theorems. The first result describes exactly the set of monomial convergence for the homogeneous polynomials on ℓ_∞.

Theorem 3.2.2. *Given $m \in \mathbb{N}$ it holds*

$$mon\mathcal{P}(^m\ell_\infty) = \ell_{\frac{2m}{m-1},\infty}.$$

Moreover, there exist an universal constant $C > 0$ such that for every $z \in \ell_{\frac{2m}{m-1},\infty}$ and every $P \in \mathcal{P}(^m\ell_\infty)$ it holds

$$\sum_{\alpha \in \Lambda(m)} |a_\alpha(P)z^\alpha| \leqslant C^m \|z\|^m_{\ell_{\frac{2m}{m-1},\infty}} \|P\|_{\mathcal{P}(^m\ell_\infty)}.$$

For the following result we will consider the set

$$B := \left\{ z \in B_{\ell_\infty} : \limsup_{n\to\infty} \frac{1}{\sqrt{\log(n)}} \left(\sum_{k=1}^n (z_k^*)^2 \right)^{1/2} < 1 \right\},$$

and its closure

$$\overline{B} = \left\{ z \in B_{\ell_\infty} : \limsup_{n\to\infty} \frac{1}{\sqrt{\log(n)}} \left(\sum_{k=1}^n (z_k^*)^2 \right)^{1/2} \leqslant 1 \right\}.$$

In the case of $H_\infty(B_{\ell_\infty})$ the following theorem is a very tight characterization of its set of monomial convergence.

Theorem 3.2.3. $B \subset monH_\infty(B_{\ell_\infty}) \subset \overline{B}$.

The last description known for these sets, in terms of classic spaces, is the set of monomial convergence for the family of homogeneous polynomials on $\ell_{r,\infty}$. In [BDS19] the authors prove that, for $r \leqslant 2 \leqslant \infty$, it holds

$$mon\mathcal{P}(^m\ell_{r,\infty}) = \ell_{\left(\frac{m-1}{2m}+\frac{1}{r}\right)^{-1},\infty}, \tag{3.7}$$

this fact was already mentioned in [DMP09] without an explicit proof and a more detailed proof may be found in Schlüters doctoral thesis [Sch15] .

For $H_\infty(B_{\ell_r})$ with $1 < r < \infty$ the lower and upper bounds for its set of monomial convergence have a "wider distance". In [DMP09, Example 4.9 (a)] the authors gave the first result looking forward a characterization the set of monomial convergence for these families, given $\varepsilon > 0$ it holds

- For $1 \leqslant r < 2$
$$\ell_1 \cap B_{\ell_r} \subset monH_\infty(B_{\ell_r}) \subset \ell_{1+\varepsilon} \cap B_{\ell_r}. \tag{3.8}$$

- For $2 \leqslant r$ and $\frac{1}{q} = \frac{1}{2} + \frac{1}{r}$
$$\ell_q \cap B_{\ell_r} \subset monH_\infty(B_{\ell_r}) \subset \ell_{q+\varepsilon} \cap B_{\ell_r}. \tag{3.9}$$

Then in [BDS19, Theorem 5.5] it appears the following refinement of the lower bounds

- For $1 < r \leqslant 2$ and $\theta > \frac{1}{2}$
$$\left(\frac{1}{n^{\frac{1}{r'}}\log(n+2)^{\frac{\theta}{r'}}}\right) \cdot B_{\ell_r} \subset monH_\infty(B_{\ell_r}). \tag{3.10}$$

- For $2 \leqslant r < \infty$ and $\theta > 0$
$$\left(\frac{1}{n^{\frac{1}{r'}}\log(n+2)^{\frac{\theta}{r'}}}\right) \cdot B_{\ell_r} \subset monH_\infty(B_{\ell_r}). \tag{3.11}$$

In the following section we present a general description of a very important and nice property that some families of holomorphic function have: the rearrangement property. We also prove that the most natural families of holomorphic functions have it.

3.3 Rearrangement families of holomorphic functions.

A very useful tool in the study of sets monomial convergence (see [BDF$^+$17]) is that usually, a sequence belongs to the set of monomial convergence if and only if its decreasing rearrangement does (see also [DGMPG08]). We isolate this property, and say in this case that $\mathcal{F}(\mathcal{R})$ is a *rearrangement family.* In [BDF$^+$17] it was proved that $H_\infty(B_{c_0})$ and $\mathcal{P}(^m c_0)$ are rearrangement families. The fact that this is also the case for ℓ_r for $1 \leqslant r < \infty$ is implicitly used in [BDS19]. Our aim now is to find other rearrangement families of holomorphic functions (compare this with [Sch15, Chapter 7] where similar results appear).

To this purpose we introduce another concept. We say a family $\mathcal{F} \subset H(\mathcal{R})$ is *linearly balanced* if $f \circ T|_{\mathcal{R}} \in \mathcal{F}$ for every $f \in \mathcal{F}$ and $T : X \to X$ linear with $\|T\| = 1$ and $T(\mathcal{R}) \subset \mathcal{R}$.

Remark 3.3.1. Rather straightforward arguments show that $H_b(X)$, $\mathcal{A}_u(B_X)$, $H_\infty(B_X)$ and $\mathcal{P}(^m X)$ for every $m \geqslant 2$ are linearly balanced families.

Theorem 3.3.2. *Let $\mathcal{R}$ be a symmetric Reinhardt domain of a symmetric Banach sequence space X and $\mathcal{F} \subset H(\mathcal{R})$ a linearly balanced family such that $\operatorname{mon}\mathcal{F} \subset c_0$, then $\mathcal{F}$ is a rearrangement family.*

We give a series of preliminary results needed for the proof of Theorem 3.3.2. Given an injective mapping $\sigma : \mathbb{N} \to \mathbb{N}$ we define two mappings in the following way. First

$$\begin{aligned} T_\sigma : \mathbb{C}^\mathbb{N} &\to \mathbb{C}^\mathbb{N} \\ x &\mapsto (x_{\sigma(k)})_{k\in\mathbb{N}} \,. \end{aligned} \tag{3.12}$$

Second, $S_\sigma : \mathbb{C}^\mathbb{N} \to \mathbb{C}^\mathbb{N}$ is defined for $x \in \mathbb{C}^\mathbb{N}$ by

$$(S_\sigma x)_k = \begin{cases} 0 & \text{if } k \notin \sigma(\mathbb{N}) \\ x_{\sigma^{-1}(k)} & \text{if } k \in \sigma(\mathbb{N}). \end{cases} \tag{3.13}$$

Both are clearly linear and $T_\sigma(S_\sigma x) = x$ for every x.

Remark 3.3.3. Let us see now how these two mappings behave with the decreasing rearrangement of a bounded sequence x. Fixed $n \in \mathbb{N}$ and $J \subset \mathbb{N}$ such that $card(J) < n$ we have

$$\sup_{\sigma(j)\in\mathbb{N}\setminus J} |x_{\sigma(j)}| = \sup_{j\in(\mathbb{N}\setminus J)\cap\sigma(\mathbb{N})} |x_j| \leqslant \sup_{j\in\mathbb{N}\setminus J} |x_j| \,.$$

Thus

$$\begin{aligned} \big(T_\sigma(x)\big)_n^* &= \inf\{ \sup_{\sigma(j)\in\mathbb{N}\setminus J} |x_{\sigma(j)}| : J \subset \mathbb{N}, \operatorname{card}(J) < n\} \\ &\leqslant \inf\{ \sup_{j\in\mathbb{N}\setminus J} |x_j| : J \subset \mathbb{N}, \operatorname{card}(J) < n\} = x_n^*. \end{aligned}$$

That is, $T_\sigma(x)^* \leqslant x^*$. A similar argument shows that $(S_\sigma x)^* = x^*$.

The following lemma shows that the restrictions of S_σ and T_σ to symmetric Banach sequence spaces are endomorphisms of norm 1.

Lemma 3.3.4. *Let X be a symmetric Banach sequence space and $\sigma : \mathbb{N} \to \mathbb{N}$ an injective mapping. Then $T_\sigma, S_\sigma : X \to X$ defined by* (3.12) *and* (3.13) *respectively are well defined, $\|T_\sigma\| = 1$ and S_σ is an isometry.*

Proof. Remark 3.3.3 together with the symmetry of the space imply that both operators are well defined, that S_σ is an isometry and $\|T_\sigma\| \leqslant 1$. The fact that $\|T_\sigma\| = 1$ follows from the equality $T_\sigma(S_\sigma x_0) = x_0$. □

Now we are able to give the proof of Theorem 3.3.2.

Proof of Theorem 3.3.2. To begin with we take $z \in mon\mathcal{F}$ and see that $z^* \in mon\mathcal{F}$. As $mon\mathcal{F} \subset c_0$ there is some injective mapping $\sigma : \mathbb{N} \to \mathbb{N}$ such that $z_k^* = |z_{\sigma(k)}|$ for every $k \in \mathbb{N}$. Observe that $|T_\sigma(z)| = z^*$. We take $f \in \mathcal{F}$, then $f \circ T_\sigma$ also belongs to $\mathcal{F}$ and what we want to see first is that, if $\alpha(\sigma) \in \mathbb{N}_0^{(\mathbb{N})}$ denotes the multi-index that fulfils $T_\sigma(z)^\alpha = z^{\alpha(\sigma)}$, then

$$c_\alpha(f) = c_{\alpha(\sigma)}(f \circ T_\sigma) \tag{3.14}$$

for every α. Take, then, some $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ and set $N = \max\{k \colon \alpha_k \neq 0\}$. On one hand we have

$$(f \circ T_\sigma)(w) = \sum_{\beta \in \mathbb{N}_0^N} c_\beta(f \circ T_\sigma) w^\beta \,,$$

for all $w \in \mathbb{C}^N \cap \mathcal{R}$. Define $M = \max\{\sigma(k) \colon k = 1, \ldots, N\}$ and note that $T_\sigma(w) \in \mathbb{C}^M \cap \mathcal{R}$. Thus

$$(f \circ T_\sigma)(w) = f(T_\sigma(w)) = \sum_{\gamma \in \mathbb{N}_0^N} c_\gamma(f) T_\sigma(w)^\gamma = \sum_{\gamma \in \mathbb{N}_0^N} c_\gamma(f) w^{\gamma(\sigma)}.$$

The uniqueness of the Taylor coefficients gives (3.14). Once we have this we obtain (recall that $f \circ T_\sigma \in \mathcal{F}$ and $z \in mon\mathcal{F}$)

$$\begin{aligned}\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f)(z^*)^\alpha| &= \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f)||(T_\sigma(z))^\alpha| \\ &= \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_{\alpha(\sigma)}(f \circ T_\sigma)||z^{\alpha(\sigma)}| \leqslant \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f \circ T_\sigma) z^\alpha| < \infty,\end{aligned}$$

which proves our claim.

For the converse, suppose $z^* \in mon\mathcal{F}$. Again, as $mon\mathcal{F} \subset c_0$, there is some injective mapping $\sigma : \mathbb{N} \to \mathbb{N}$ such that $z_k^* = |z_{\sigma(k)}|$ for every $k \in \mathbb{N}$. Now it will be useful to notice $|z| = S_\sigma(z^*)$. Given $f \in \mathcal{F}$ we have

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f) z^\alpha| = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f)||(S_\sigma(z^*))^\alpha|. \tag{3.15}$$

Besides,

$$\sum_{\alpha\in\mathbb{N}_0^N} c_\alpha(f\circ S_\sigma)w^\alpha = f(S_\sigma(w)) = \sum_{\alpha\in\mathbb{N}_0^N} c_\alpha(f)S_\sigma(w)^\alpha = \sum_{\alpha\in\mathbb{N}_0^N} c_\alpha(f)S_\sigma(w)^\alpha.$$

Observe that for $\alpha \in \mathbb{N}^{(\mathbb{N})}$, if there is $k \in \mathbb{N}\backslash\sigma(\mathbb{N})$ such that $\alpha_k \neq 0$ then $S_\sigma(w)^\alpha = 0$, otherwise we define $\alpha(\sigma^{-1}) \in \mathbb{N}^{(\mathbb{N})}$ as the only multi-index which fulfils $S_\sigma(w)^\alpha = w^{\alpha(\sigma^{-1})}$. By the uniqueness of the coefficients of the Taylor expansion for $f\circ S_\sigma : \mathbb{C}^N \to \mathbb{C}$ it follows

$$c_\alpha(f)S_\sigma(z^*)^\alpha = \begin{cases} 0 & \text{if there is } k\notin\sigma(\mathbb{N}) \text{ such that } \alpha_k\neq 0\\ c_{\alpha(\sigma^{-1})}(f\circ S_\sigma)(z^*)^{\alpha(\sigma^{-1})} & \text{otherwise,}\end{cases}$$

then

$$\begin{aligned}\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f)z^\alpha| &= \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f)||(S_\sigma(z^*))^\alpha|\\ &= \sum_{\alpha\in(\sigma(\mathbb{N})\cup\{0\})^{(\mathbb{N})}} \left|c_{\alpha(\sigma^{-1})}(f\circ S_\sigma)(z^*)^{\alpha(\sigma^{-1})}\right|\\ &\leqslant \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f\circ S_\sigma)||(z^*)^\alpha| < \infty,\end{aligned}$$

as we wanted. □

Remark 3.3.5. Let $\mathcal{R}$ be a symmetric Reinhardt domain in a Banach sequence space X and consider a family of homolorphic functions $\mathcal{F} \subset H(\mathcal{R})$ such that for some $m \geqslant 2$ the space $\mathcal{P}(^mX)$ lies inside $\mathcal{F}$. Then, as $X \subset \ell_\infty$ continuously we have $\mathcal{P}(^m\ell_\infty) \subset \mathcal{P}(^mX) \subset \mathcal{F}$. With this, Theorem 3.2.2 yields

$$mon\mathcal{F} \subset mon\mathcal{P}(^m\ell_\infty) = \ell_{\frac{2m}{m-1},\infty} \subset c_0.$$

Corollary 3.3.6. *For every symmetric Banach sequence space X the families of holomorphic functions $H_b(X), \mathcal{A}_u(B_X), H_\infty(B_X)$ and $\mathcal{P}(^mX)$ with $m \geqslant 2$ are rearrangement families.*

Proof. Each of these families satisfies the condition in Remark 3.3.5. Then Remark 3.3.1 and Theorem 3.3.2 give the conclusion. □

3.4 The set of monomial convergence of $\mathcal{P}(^m\ell_r)$

We now turn our attention to the set of monomial convergence of the homogeneous polynomials. As we have already mentioned for $2 \leqslant r \leqslant \infty$ and $m \geqslant 2$ it holds

$$mon\mathcal{P}(^m\ell_{r,\infty}) = \ell_{q,\infty},$$

with $q = q(r) = \left(\frac{m-1}{2m} + \frac{1}{r}\right)^{-1}$. As the natural inclusion $\ell_r \hookrightarrow \ell_{r,\infty}$ is continuous it holds $\mathcal{P}(^m\ell_{r,\infty}) \subset \mathcal{P}(^m\ell_r)$ and then

$$mon\mathcal{P}(^m\ell_r) \subset mon\mathcal{P}(^m\ell_{r,\infty}) = \ell_{q,\infty}.$$

Also using Theorem 3.2.2 and Lemma 3.1.6 with $\mathcal{R} = \ell_r$ and $\mathcal{F}(\mathcal{R}) = \mathcal{P}(^m\ell_r)$ we have

$$\ell_r \cdot \ell_{\frac{2m}{m-1},\infty} = \ell_r \cdot mon\mathcal{P}(^m\ell_\infty) \subset \ell_r \cdot [\mathcal{P}(^m\ell_r)]_\infty \subset mon\mathcal{P}(^m\ell_r). \tag{3.16}$$

For $1 < r \leqslant 2$ and $m \geqslant 2$, define $q = (mr')' = \frac{mr}{r(m-1)+1}$. In [DMP09, Example 4.6] the authors prove that

$$\ell_{q-\varepsilon} \subset mon\mathcal{P}(^m\ell_r) \subset \ell_{q,\infty}, \tag{3.17}$$

for every $\varepsilon > 0$. Our aim now is to tighten this lower bound. In [DMP09] the authors conjecture that

$$\ell_q \subset mon\mathcal{P}(^m\ell_r), \tag{3.18}$$

where $q = (mr')'$ as before. In the following theorem we present the first result improving the lower bound in (3.17) and proving that conjecture. Later in Chapter 8 we give a better result for the lower bound with spaces that get closer to $\ell_{q,\infty}$ as the homogeneity degree goes to infinity.

Theorem 3.4.1. *For each $1 < r \leqslant 2$, there exists $d_r > 1$ such that for each m and n, every $P \in \mathcal{P}(^m\mathbb{C}^n)$ and all $z \in \mathbb{C}^n$*

$$\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}| \leqslant m^{d_r} \|P\|_{\mathcal{P}(^m\ell_r^n)} \|z\|_{\ell_q^n}^m \,, \tag{3.19}$$

where $q := (mr')'$.

We will first use Theorem 3.4.1 to show the conjecture previously mentioned is true and then using a technical lemma we will be able to prove it.

Corollary 3.4.2. *Given $1 < r \leqslant 2$, $m \geqslant 2$ and $q = (mr')'$ it holds*

$$\ell_q \subset mon\mathcal{P}(^m\ell_r).$$

Proof. Fix $z \in \ell_q$ and take $P \in \mathcal{P}(^m\ell_r)$. Given $n \in \mathbb{N}$ consider $P_n = P \circ \iota_n \in \mathcal{P}(^m\mathbb{C}^n)$ and $\pi_n z \in \mathbb{C}^n$ where π_n and ι_n are the natural projection and inclusion defined in (1.3) and (1.4) respectively. Notice that $\|P_n\|_{\mathcal{P}(^m\ell_r^n)} \leqslant \|P\|_{\mathcal{P}(^m\ell_r)}$ and $\|\pi_n(z)\|_{\ell_q^n} \leqslant \|z\|_{\ell_q}$. Also by the construction we did of the monomial expansion of an holomorphic function we have, for every $\mathbf{j} \in \mathcal{J}(m,n)$ that $c_{\mathbf{j}}(P_n) = c_{\mathbf{j}}(P)$.

Thanks to Theorem 3.4.1 we have

$$\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P)(\pi_n z)_{\mathbf{j}}| \leqslant m^{d_r} \|P_n\|_{\mathcal{P}(^m\ell_r^n)} \|\pi_n z\|_{\ell_q^n}^m \leqslant m^{d_r} \|P\|_{\mathcal{P}(^m\ell_r)} \|z\|_{\ell_q}^m < \infty$$

taking the limit $n \to \infty$ it follows

$$\sum_{\mathbf{j}\in\mathcal{J}(m)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}| \leqslant m^{d_r} \|P\|_{\mathcal{P}(^m\ell_r)} \|z\|_{\ell_q}^m < \infty.$$

As this holds for every $P \in \mathcal{P}(^m\ell_r)$ then $z \in mon\mathcal{P}(^m\ell_r)$. □

To prove Theorem 3.4.1 we need the following technical lemma.

Lemma 3.4.3. *Let $r>1$. There exists $C_r>0$ such that, for every m,*

$$\sup\Big\{\frac{m^{m/r}}{m!}\frac{n_1!}{n_1^{n_1/r}}\cdots\frac{n_k!}{n_k^{n_k/r}} : k\in\mathbb{N}, n_1,\dots,n_k\in\mathbb{N}\backslash\{0\}, n_1+\cdots+n_k=m\Big\}\leqslant C_r m^{\frac{e^{\frac{1}{r-1}}-1}{2}}\,.$$

Proof. We proceed by induction on m. The statement is trivially satisfied for $m=2$ and we assume it holds for $m-1$. Fix then k and choose $n_1,\dots,n_k\in\mathbb{N}$, all non-zero, such that $n_1+\cdots+n_k=m$. We may assume $n_1\geqslant\cdots\geqslant n_k\geqslant 1$. We consider two possible cases. First, if $k<e^{\frac{1}{r-1}}$ Stirling inequality in (2.16) and the fact that $n_j\leqslant m$ for every j yield

$$\begin{aligned}\frac{m^{m/r}}{m!}\frac{n_1!}{n_1^{n_1/r}}\cdots\frac{n_k!}{n_k^{n_k/r}} &\leqslant \frac{1}{\sqrt{2\pi m}}\frac{e^m}{m^{m/r'}}\prod_{j=1}^{k}\frac{\sqrt{2\pi n_j}n_j^{n_j/r'}e^{1/(12n_j)}}{e^{n_j}}\\ &\leqslant (2\pi)^{\frac{k-1}{2}}e^{\sum_{j=1}^k\frac{1}{12n_j}}\Big(\frac{n_1^{n_1}\cdots n_k^{n_k}}{m^m}\Big)^{\frac{1}{r'}}\Big(\frac{n_1\cdots n_k}{m}\Big)^{\frac12}\leqslant (2\pi)^{\frac{k-1}{2}}e^{\sum_{j=1}^k\frac{1}{12j}}m^{\frac{k-1}{2}}\\ &\leqslant (2\pi)^{\frac{e^{\frac{1}{r-1}}-1}{2}}e^{\frac{r}{12(r-1)}}m^{\frac{e^{\frac{1}{r-1}}-1}{2}}\,.\end{aligned}$$

On the other hand, if $k\geqslant e^{\frac{1}{r-1}}$ we have

$$\frac{m^{m/r}}{m!}\frac{n_1!}{n_1^{n_1/r}}\cdots\frac{n_k!}{n_k^{n_k/r}}=\Big(\frac{m}{m-1}\Big)^{\frac{m-1}{r}}\frac{1}{m^{1/r'}}\frac{(m-1)^{(m-1)/r}}{(m-1)!}\frac{n_1!}{n_1^{n_1/r}}\cdots\frac{n_{k-1}!}{n_{k-1}^{n_{k-1}/r}}\frac{n_k!}{n_k^{n_k/r}}\,. \tag{3.20}$$

If $n_k=1$ then $n_1+\cdots+n_{k-1}=m-1$ and we may use the induction hypothesis and the fact that $k\leqslant m$ to have

$$\begin{aligned}\frac{m^{m/r}}{m!}\frac{n_1!}{n_1^{n_1/r}}\cdots\frac{n_k!}{n_k^{n_k/r}}&\leqslant\Big(\frac{m}{m-1}\Big)^{\frac{m-1}{r}}\frac{1}{k^{1/r'}}C_r(m-1)^{\frac{e^{\frac{1}{r-1}}-1}{2}}\\ &\leqslant C_r e^{1/r}\frac{1}{e^{\frac{1}{(r-1)r'}}}(m-1)^{\frac{e^{\frac{1}{r-1}}-1}{2}}\leqslant C_r m^{\frac{e^{\frac{1}{r-1}}-1}{2}}\,.\end{aligned}$$

Finally, if $n_k>1$ then

$$\frac{(n_k-1)^{\frac{n_k-1}{r'}}n_k}{n_k^{n_k/r}}=\Big(\frac{n_k-1}{n_k}\Big)^{\frac{n_k-1}{r'}}n_k^{\frac{1}{r'}}\leqslant n_k^{\frac{1}{r'}}\,.$$

We may use again the induction hypothesis and the fact that $n_k\leqslant m/k$ to obtain from (3.20)

$$\begin{aligned}\frac{m^{m/r}}{m!}\frac{n_1!}{n_1^{n_1/r}}\cdots\frac{n_k!}{n_k^{n_k/r}}&\leqslant\Big(\frac{m}{m-1}\Big)^{\frac{m-1}{r}}\Big(\frac{n_k}{m}\Big)^{1/r'}C_r(m-1)^{\frac{e^{\frac{1}{r-1}}-1}{2}}\\ &\leqslant\Big(\frac{m}{m-1}\Big)^{\frac{m-1}{r}}\frac{1}{k^{1/r'}}C_r(m-1)^{\frac{e^{\frac{1}{r-1}}-1}{2}}.\end{aligned}$$

From here we conclude as in the previous case. ∎

Proof of Theorem 3.4.1. Clearly it is enough to show (3.19) and, by the Corollary 3.3.6 we may assume without loss of generality $z = z^*$. First of all, by Hölder inequality we have

$$\sum_{1\leqslant j_1\leqslant\cdots\leqslant j_m\leqslant n} |c_{\mathbf{j}}(P) z_{j_1}\dots z_{j_{m-1}} z_{j_m}| = \sum_{1\leqslant j_1\leqslant\cdots\leqslant j_{m-1}\leqslant n} |z_{j_1}\dots z_{j_{m-1}}| \sum_{j_m=j_{m-1}}^{n} |c_{\mathbf{j}}(P) z_{j_m}|$$
$$\leqslant \sum_{1\leqslant j_1\leqslant\cdots\leqslant j_{m-1}\leqslant n} |z_{j_1}\dots z_{j_{m-1}}| \Big(\sum_{j_m=j_{m-1}}^{n} |c_{\mathbf{j}}(P)|^{r'}\Big)^{\frac{1}{r'}} \Big(\sum_{j_m=j_{m-1}}^{n} |z_{j_m}^r|\Big)^{\frac{1}{r}}$$

Using *BDS inequality* in Theorem 2.1.7 together with the fact that for every $(\mathbf{i},k) \in \mathcal{J}(m-1,n)$ we have $\big(\frac{(m-1)^{m-1}}{\alpha(\mathbf{i},k)^{\alpha(\mathbf{i},k)}}\big) \leqslant e(m-1)\big(\frac{(m-2)^{m-2}}{\alpha(\mathbf{i})^{\alpha(\mathbf{i})}}\big)$ we obtain

$$\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}|$$
$$\leqslant e^{1+\frac{1}{r}}(m-1)^{\frac{1}{r}} m \|P\|_{\mathcal{P}(^m\ell_r^n)} \sum_{j_{m-1}=1}^{n} |z_{j_{m-1}}| \sum_{\mathbf{i}\in\mathcal{J}(m-2,j_{m-1})} |z_{\mathbf{i}}| \Big(\frac{(m-2)^{m-2}}{\alpha(\mathbf{i})^{\alpha(\mathbf{i})}}\Big)^{\frac{1}{r}} \Big(\sum_{j_m=j_{m-1}}^{n} |z_{j_m}|^r\Big)^{\frac{1}{r}}$$

For each fixed $1 \leqslant k \leqslant n$, using $|[\mathbf{j}]|$ and Lemma 3.4.3 (we write $a_r = \frac{e^{\frac{1}{r-1}}-1}{2}$) and the fact that $q \leqslant r$, we have

$$|z_k| \sum_{\mathbf{i}\in\mathcal{J}(m-2,k)} |z_{\mathbf{i}}| \Big(\frac{(m-2)^{m-2}}{\alpha(\mathbf{i})^{\alpha(\mathbf{i})}}\Big)^{\frac{1}{r}} \Big(\sum_{j=k}^{n} |z_j|^r\Big)^{\frac{1}{r}}$$
$$\leqslant |z_k| \sum_{\mathbf{i}\in\mathcal{J}(m-2,k)} |z_{\mathbf{i}}||\mathbf{i}| \frac{(m-2)^{(m-2)/r}}{\alpha(\mathbf{i})^{\alpha(\mathbf{i})/r}|\mathbf{i}|} \Big(|z_k|^{r-q} \sum_{j=k}^{n} |z_j|^q\Big)^{\frac{1}{r}}$$
$$= C_r(m-2)^{a_r} |z_k|^{2-\frac{q}{r}} \sum_{i_1,\dots,i_{m-2}=1}^{k} |z_{i_1}\cdots z_{i_{m-2}}| \Big(\sum_{j=k}^{n} |z_j|^q\Big)^{\frac{1}{r}}$$
$$= C_r(m-2)^{a_r} \|z\|_{\ell_q}^{\frac{q}{r}} |z_k|^{2-\frac{q}{r}} \Big(\sum_{i=1}^{k} |z_i|\Big)^{m-2}$$
$$\leqslant C_r(m-2)^{a_r} \|z\|_{\ell_q^n}^{\frac{q}{r}+m-2} |z_k|^{2-\frac{q}{r}} k^{\frac{m-2}{q'}}.$$

On the other hand, since $2-\frac{q}{r} \geqslant q$ for $m \geqslant 2$, it follows

$$\sum_{k=1}^{n} |z_k|^{2-\frac{q}{r}} k^{\frac{m-2}{q'}} = \|z\|_{\ell^n_{q,2-q/r}}^{2-\frac{q}{r}} \leqslant \|z\|_{\ell_q^n}^{2-\frac{q}{r}}$$

This altogether gives

$$\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}| \leqslant K_r m(m-1)^{\frac{1}{r}} (m-2)^{a_r} \|P\|_{\mathcal{P}(^m\ell_r^n)} \|z\|_{\ell_q^n}^m . \qquad \square$$

Chapter 4

Unconditionality in spaces of polynomials

In this chapter we present the unconditionality on the context of spaces of polynomials. The unconditional constant of the spaces of homogeneous polynomials and their dependence on the number of variables and the degree of homogeneity will be crucial to study the Bohr radius, and the sets of monomial convergence in the following chapters.

Recall that a Schauder basis $(b_n)_{n\in\mathbb{N}}$ of a Banach space X is unconditional if given $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ and $x = \sum_{n\geqslant 1} a_n b_n \in X$ we have $\sum_{n\geqslant 1} a_{\sigma(n)} b_n \in X$ for every $\sigma \in S_{\mathbb{N}}$. This is a very useful and qualitative description for the notion of unconditionality for a basis but it only makes sense to study this on an infinite dimensional space, as it is trivially fulfilled for every basis on the finite dimensional context. On the other hand this notion is equivalent to the existence of some constant $K > 0$ such that for every $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ and every $(\varepsilon_n)_{n\in\mathbb{N}} \subset \mathbb{T}^{\mathbb{N}}$ it holds

$$\left\| \sum_{n\geqslant 1} \varepsilon_n a_n b_n \right\|_X \leqslant K \left\| \sum_{n\geqslant 1} a_n b_n \right\|_X . \tag{4.1}$$

For a fixed basis we may think of the best possible constant in (4.1). This last way of describing the unconditionality of a basis gives a more quantitative approach. This approach makes sense in the case of a finite dimensional Banach space, and it will be very fruitful for $\mathcal{P}(^m\mathbb{C}^n)$.

4.1 Mixed unconditionality and the monomial basis

Now we define a slightly more general concept for spaces of m-homogeneous polynomials on n complex variables. In the vector space $\mathcal{P}(^m\mathbb{C}^n)$ there are many natural norms, in particular the family of uniform norms for ℓ_p^n and ℓ_q^n are different when $p \neq q$. We will define the concept of mixed unconditionality inspired in (4.1) but allowing the norms taken in the right and the left side of the inequality to be different. The following definition makes the last statement precise.

Definition 4.1.1. *Let $(P_i)_{i\in\Lambda}$ be a Schauder basis of $\mathcal{P}({}^m\mathbb{C}^n)$. For $1 \leqslant p,q \leqslant \infty$ and $n,m \in \mathbb{N}$ we define $\chi_{p,q}((P_i)_{i\in\Lambda}) = \chi_{p,q}((P_i)_{i\in\Lambda};\mathcal{P}({}^m\mathbb{C}^n))$ as the best constant $C > 0$ such that*

$$\left\| \sum_{i\in\Lambda} \theta_i c_i P_i \right\|_{\mathcal{P}({}^m\ell_q^n)} \leqslant C \left\| \sum_{i\in\Lambda} c_i P_i \right\|_{\mathcal{P}({}^m\ell_p^n)}, \tag{4.2}$$

for every $P = \sum_{i\in\Lambda} c_i P_i \in \mathcal{P}({}^m\mathbb{C}^n)$ and every choice of complex numbers $(\theta_i)_{i\in\Lambda}$ of modulus one.

The (p,q)-mixed unconditional constant *of $\mathcal{P}({}^m\mathbb{C}^n)$ is defined as*

$$\chi_{p,q}(\mathcal{P}({}^m\mathbb{C}^n)) := \inf\{\chi_{p,q}((P_i)_{i\in\Lambda}) : (P_i)_{i\in\Lambda} \text{ basis for } \mathcal{P}({}^m\mathbb{C}^n)\}.$$

This idea was introduced by Defant, Maestre and Prengel in [DMP09, Section 5]. Notice that for $p = q$ the concept of mixed unconditionality coincides with the notion of unconditionality over $\mathcal{P}({}^m\ell_p^n)$ as it is defined for Banach spaces. It will be interesting and natural to study the mixed unconditional constant of the spaces $\mathcal{P}({}^m\mathbb{C}^n)$ as well as the particular constant for the monomial basis $(z_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$ which can also be written as $(z^\alpha)_{\alpha\in\Lambda(m,n)}$. It will become clear in the following chapters the deep connection between $\chi_{p,q}((z_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)};\mathcal{P}({}^m\mathbb{C}^n))$, the mixed Bohr radius and the sets of monomial convergence.

The following result shows that, in order to study the asymptotic behavior of the mixed unconditional constants of $\mathcal{P}({}^m\mathbb{C}^n)$, it is enough to understand what happens with the monomial basis $(z_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$. These can be seen as a sort extension of a result of Pisier and Schütt [Pis78, Sch78] (see also [DDGM01, DF11, CG11]).

Theorem 4.1.2. *Given $m,n \in \mathbb{N}$ and $1 \leqslant p,q \leqslant \infty$ we have the following relation*

$$\chi_{p,q}(\mathcal{P}({}^m\mathbb{C}^n)) \leqslant \chi_{p,q}\big((z_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}\big) \leqslant 2^m \chi_{p,q}(\mathcal{P}({}^m\mathbb{C}^n)).$$

Our proof relies on Szarek's approach [Sza81] combined with the following inequality due to Bayart [Bay02] (see also [Wei80, DM15]).

Lemma 4.1.3 (Bayart's inequality)**.** *Let $P(z) = \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}} z_{\mathbf{j}}$ be an m-homogeneous polynomial in n-variables. Then*

$$\Big(\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}|^2 \Big)^{1/2} \leqslant 2^{m/2} \int_{\mathbb{T}^n} |P(w)| dw, \tag{4.3}$$

where $\mathbb{T}^n$ stands for the n-dimensional torus and dw is the normalized Lebesgue measure on $\mathbb{T}^n$.

Before giving the proof we define the following operator. For any $w = (w_1 \ldots, w_n) \in \mathbb{T}^n$ and any $1 \leqslant p \leqslant \infty$ we define the operator

$$\begin{aligned} T_w^p : \mathcal{P}({}^m\ell_p^n) &\longrightarrow \mathcal{P}({}^m\ell_p^n) \\ \sum_{\mathbf{j}\in\mathcal{J}(m,n)} a_{\mathbf{j}} z_{\mathbf{j}} &\longmapsto \sum_{\mathbf{j}\in\mathcal{J}(m,n)} a_{\mathbf{j}} z_{\mathbf{j}} w_{\mathbf{j}}, \end{aligned}$$

which clearly has norm one.

We also need the following alternative characterization of the mixed unconditional constant for a given basis.

Lemma 4.1.4. *Let $(P_i)_{i\in\Lambda}$ be a basis for $\mathcal{P}(^m\mathbb{C}^n)$ and $(P_i')_{i\in\Lambda}$ its dual basis (i.e., $\langle P_i', P_k\rangle = \delta_{i.k}$). For $1 \leqslant q, p \leqslant \infty$ and $n, m \in \mathbb{N}$, $\chi_{p,q}((P_i)_{i\in\Lambda})$ is exactly the best constant $C > 0$ such that*

$$\sum_{i\in\Lambda} |\langle P_i', Q\rangle\langle Q', P_i\rangle| \leqslant C\|Q\|_{\mathcal{P}(^m\ell_p^n)}\|Q'\|_{\mathcal{P}(^m\ell_q^n)'}, \tag{4.4}$$

for every $Q \in \mathcal{P}(^m\mathbb{C}^n)$ and $Q' \in \mathcal{P}(^m\mathbb{C}^n)'$.

Before proving Lemma 4.1.4 notice that given $Q = \mathcal{P}(^m\mathbb{C}^n)$ and $(P_i)_{i\in\Lambda}$ a basis for $\mathcal{P}(^m\mathbb{C}^n)$ with dual basis $(P_i')_{i\in\Lambda}$ it holds

$$Q = \sum_{i\in\Lambda}\langle P_i', Q\rangle P_i.$$

Proof. Let us name C_1 to best constant fullfilling equation (4.4). We want to prove $C_1 = \chi_{p,q}((P_i)_{i\in\Lambda})$. We will first prove $C_1 \leqslant \chi_{p,q}((P_i)_{i\in\Lambda})$.

Fix $Q \in \mathcal{P}(^m\mathbb{C}^n)$ and $Q' \in \mathcal{P}(^m\mathbb{C}^n)'$ and, for $i \in \Lambda$, let θ_i be the sign of $\langle Q', P_i\rangle\langle P_i', Q\rangle$ then we have

$$\begin{aligned}
\sum_{i\in\Lambda} |\langle P_i', Q\rangle\langle Q', P_i\rangle| &= \sum_{i\in\Lambda}\theta_i\langle P_i', Q\rangle\langle Q', P_i\rangle \\
&\leqslant \left\|\sum_{i\in\Lambda}\theta_i\langle P_i', Q\rangle P_i\right\|_{\mathcal{P}(^m\ell_q^n)}\|Q'\|_{\mathcal{P}(^m\ell_q^n)'} \\
&\leqslant \chi_{p,q}((P_i)_{i\in\Lambda})\left\|\sum_{i\in\Lambda}\langle P_i', Q\rangle P_i\right\|_{\mathcal{P}(^m\ell_p^n)}\|Q'\|_{\mathcal{P}(^m\ell_q^n)'} \\
&= \chi_{p,q}((P_i)_{i\in\Lambda})\|Q\|_{\mathcal{P}(^m\ell_p^n)}\|Q'\|_{\mathcal{P}(^m\ell_q^n)'}.
\end{aligned}$$

Then $\chi_{p,q}((P_i)_{i\in\Lambda})$ meets the inequality in (4.4) for every pair $Q \in \mathcal{P}(^m\mathbb{C}^n)$ and $Q' \in \mathcal{P}(^m\mathbb{C}^n)'$, by the minimality of C_1 it holds $C_1 \leqslant \chi_{p,q}((P_i)_{i\in\Lambda})$.

On the other hand, let $(\theta_i)_{i\in\Lambda} \subset \mathbb{T}$ and $(c_i)_{i\in\Lambda} \subset \mathbb{C}$. Take $Q = \sum_{i\in\Lambda} c_iP_i \in \mathcal{P}(^m\mathbb{C}^n)$ and

$Q' \in \mathcal{P}({}^m\mathbb{C}^n)'$ such that $\|\sum_{i\in\Lambda}\theta_i c_i P_i\|_{\mathcal{P}({}^m\ell_q^n)} = Q'(\sum_{i\in\Lambda}\theta_i c_i P_i)$ and $\|Q'\|_{\mathcal{P}({}^m\ell_p^n)'} = 1$ then

$$\begin{aligned}
\left\|\sum_{i\in\Lambda}\theta_i c_i P_i\right\|_{\mathcal{P}({}^m\ell_q^n)} &= Q'\left(\sum_{i\in\Lambda}\theta_i c_i P_i\right)\\
&= \sum_{i\in\Lambda}\theta_i c_i\langle Q', P_i\rangle\\
&\leqslant \sum_{i\in\Lambda}|\theta_i c_i\langle Q', P_i\rangle|\\
&\leqslant \sum_{i\in\Lambda}|\langle P_i', Q\rangle\langle Q', P_i\rangle|\\
&\leqslant C_1\|Q\|_{\mathcal{P}({}^m\ell_p^n)}\|Q'\|_{\mathcal{P}({}^m\ell_q^n)'}\\
&= C_1\left\|\sum_{i\in\Lambda}c_i P_i\right\|_{\mathcal{P}({}^m\ell_p^n)},
\end{aligned}$$

then C_1 satisfies inequality in (4.2) and by the minimality of $\chi_{p,q}((P_i)_{i\in\Lambda})$ we have

$$\chi_{p,q}((P_i)_{i\in\Lambda}) \leqslant C_1. \qquad \square$$

Now we are able to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. Let $(P_i)_{i\in\Lambda}$ be a basis for $\mathcal{P}({}^m\mathbb{C}^n)$ and $(P_i')_{i\in\Lambda}$ its dual basis. Consider $Q \in \mathcal{P}({}^m\mathbb{C}^n)$ and $Q' \in \mathcal{P}({}^m\mathbb{C}^n)'$. Since $1 = |\langle z_{\mathbf{j}}', z_{\mathbf{j}}\rangle| = |\sum_{i\in\Lambda}\langle z_{\mathbf{j}}', P_i\rangle\langle P_i', z_{\mathbf{j}}\rangle|$, we have

$$\begin{aligned}
\sum_{\mathbf{j}\in\mathcal{J}(m,n)}|\langle Q', z_{\mathbf{j}}\rangle\langle z_{\mathbf{j}}', Q\rangle| &= \sum_{\mathbf{j}\in\mathcal{J}(m,n)}|\langle Q', z_{\mathbf{j}}\rangle\langle z_{\mathbf{j}}', Q\rangle||\sum_{i\in\Lambda}\langle z_{\mathbf{j}}', P_i\rangle\langle P_i', z_{\mathbf{j}}\rangle|\\
&\leqslant \sum_{i\in\Lambda}\sum_{\mathbf{j}\in\mathcal{J}(m,n)}|\langle Q', z_{\mathbf{j}}\rangle\langle z_{\mathbf{j}}', Q\rangle\langle z_{\mathbf{j}}', P_i\rangle\langle P_i', z_{\mathbf{j}}\rangle|\\
&\leqslant \sum_{i\in\Lambda}\Big(\sum_{\mathbf{j}\in\mathcal{J}(m,n)}|\langle Q', z_{\mathbf{j}}\rangle\langle z_{\mathbf{j}}', P_i\rangle|^2\Big)^{\frac{1}{2}}\Big(\sum_{\mathbf{j}\in\mathcal{J}(m,n)}|\langle z_{\mathbf{j}}', Q\rangle\langle P_i', z_{\mathbf{j}}\rangle|^2\Big)^{\frac{1}{2}}\\
&\leqslant \sum_{i\in\Lambda}2^{m/2}\int_{\mathbb{T}^n}|\sum_{\mathbf{j}\in\mathcal{J}(m,n)}\langle Q', z_{\mathbf{j}}\rangle\langle z_{\mathbf{j}}', P_i\rangle w_{\mathbf{j}}|dw \;\cdot\; 2^{m/2}\int_{\mathbb{T}^n}|\sum_{\mathbf{j}\in\mathcal{J}(m,n)}\langle z_{\mathbf{j}}', Q\rangle\langle P_i', z_{\mathbf{j}}\rangle\tilde{w}_{\mathbf{j}}|d\tilde{w}\\
&= \sum_{i\in\Lambda}2^m\int_{\mathbb{T}^n}|\langle(T_w^q)^*(Q'), P_i\rangle|dw \;\cdot\; \int_{\mathbb{T}^n}|\langle P_i', T_{\tilde{w}}^p(Q)\rangle|d\tilde{w}\\
&= 2^m\int_{\mathbb{T}^n\times\mathbb{T}^n}\sum_{i\in\Lambda}|\langle(T_w^q)^*(Q'), P_i\rangle\langle P_i', T_{\tilde{w}}^p(Q)\rangle|dwd\tilde{w}\\
&\leqslant 2^m\int_{\mathbb{T}^n\times\mathbb{T}^n}\chi_{p,q}((P_i)_{i\in\Lambda})\|(T_w^q)^*(Q')\|_{\mathcal{P}({}^m\ell_q^n)'}\|T_{\tilde{w}}^p(Q)\|_{\mathcal{P}({}^m\ell_p^n)}dwd\tilde{w}\\
&\leqslant 2^m\chi_{p,q}((P_i)_{i\in\Lambda})\|Q'\|_{\mathcal{P}({}^m\ell_q^n)'}\|Q\|_{\mathcal{P}({}^m\ell_p^n)},
\end{aligned}$$

where we applied Cauchy-Schwarz for the second inequality, Bayart's inequality (4.3) for the third one and Lemma 4.1.4 for the basis (P_i) for the next to last inequality. Using Lemma 4.1.4 again but for the monomial basis $(z_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$ we have that

$$\chi_{p,q}\big((z_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}\big) \leqslant 2^m \chi_{p,q}((P_i)_{i\in\Lambda}).$$

Since $(P_i)_{i\in\Lambda}$ is an arbitrary basis of $\mathcal{P}(^m\mathbb{C}^n)$ we have

$$\chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \leqslant \chi_{p,q}\big((z_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}\big) \leqslant 2^m \chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)),$$

which concludes the proof. □

4.2 Link with monomial convergence

There is a very deep link between the notions of mixed unconditionality in spaces of polynomials and monomial convergence. This connection was developed in [DMP09] and to have a full version of it (which will be necessary in the future) we need to generalize the concept of mixed unconditionality for the monomial basis.

Definition 4.2.1. *Fix $n, m \in \mathbb{N}$ and let $X_n = (\mathbb{C}^n, \|\cdot\|_{X_n})$ and $Y_n = (\mathbb{C}^n, \|\cdot\|_{Y_n})$ be two n dimensional Banach spaces over $\mathbb{C}$. We define $\chi_M(\mathcal{P}(^mX_n), \mathcal{P}(^mY_n))$ as the best constant $C > 0$ such that*

$$\Big\| \sum_{\alpha\in\Lambda(m,n)} \theta_\alpha a_\alpha z^\alpha \Big\|_{\mathcal{P}(^mY_n)} \leqslant C \Big\| \sum_{\alpha\in\Lambda(m,n)} a_\alpha z^\alpha \Big\|_{\mathcal{P}(^mX_n)}, \tag{4.5}$$

for every $(a_\alpha)_{\alpha\in\Lambda(m,n)} \subset \mathbb{C}$ and every choice of complex numbers $(\theta_\alpha)_{\alpha\in\Lambda(m,n)}$ of modulus one.

Observe that for $1 \leqslant p, q \leqslant \infty$, if $X_n = \ell_p^n$ and $Y_n = \ell_q^n$, it holds

$$\chi_M(\mathcal{P}(^mX_n), \mathcal{P}(^mY_n)) = \chi_{p,q}\big((z^\alpha)_{\alpha\in\Lambda(m,n)}\big).$$

We will present below two very important tools that show the intimate relationship between the set of monomial convergence and the mixed unconditionality for the monomial basis. Both results appear in [DMP09] and have more equivalences on their statements.

Theorem 4.2.2 ([DMP09]Theorem 5.1)**.** *Let X and Y be Banach sequence spaces, $\mathcal{R} \subset X$ a bounded Reinhardt domain and $\mathcal{F}(\mathcal{R})$ a closed subfamily of $H_\infty(\mathcal{R})$ which contains all the polynomials. The following are equivalent:*

(1) $rB_Y \subset mon\mathcal{F}(\mathcal{R})$ for some $r > 0$.

(2) There is a constant $C > 0$ depending only on X, Y such that for every $m \in \mathbb{N}$

$$\sup_{n\in\mathbb{N}} \chi_M(\mathcal{P}(^mX_n), \mathcal{P}(^mY_n)) \leqslant C^m.$$

Recall that given X a Banach sequence space we denote $X_n = \pi_n(X) \subset \mathbb{C}^n$ to its n-dimensional projection endowed with the induced norm.

Theorem 4.2.3. *Let X and Y be Banach sequence spaces and $m \geqslant 2$. Then the following statements are equivalent:*

(1) $Y \subset mon\mathcal{P}(^m X)$.

(2) $\sup_{n\in\mathbb{N}} \chi_M(\mathcal{P}(^m X_n), \mathcal{P}(^m Y_n)) < \infty$.

Given X and Y Banach sequence spaces we say the asymptotic behaviour of the monomial mixed unconditional constant $\chi_M(\mathcal{P}(^m X_n), \mathcal{P}(^m Y_n))$ is *hypercontractive* on the homogeneity degree m whenever there is some $C(X, Y, n) > 0$ which does not depend on m such that

$$\chi_M(\mathcal{P}(^m X_n), \mathcal{P}(^m Y_n)) \leqslant C(X, Y, n)^m, \tag{4.6}$$

for every $m, n \in \mathbb{N}$.

Notice that one of the equivalences in Theorem 4.2.2 is that the asymptotic behaviour of $\chi_M(\mathcal{P}(^m X_n), \mathcal{P}(^m Y_n))$ is trivial on n and hypercontrative on m. On the other hand Theorem 4.2.3 gives the trivial asymptotic behaviour of that constant on the number of variables but it does not give hypercontractivity as one of the equivalences.

Remark 4.2.4. Given $1 < p \leqslant 2$ thanks to Corollary 3.4.2 and Theorem 4.2.3 there is $C(p, m) > 0$ such that for any $n, m \in \mathbb{N}$ it holds

$$\chi_{p,q}((z^\alpha)_{\alpha\in\Lambda(m,n)}) \leqslant C(p, m) < \infty,$$

where $q = q(p) = (mp')'$.

Then, fixed m, the asymptotic behaviour of $\chi_M(\mathcal{P}(^m \ell_p^n), \mathcal{P}(^m \ell_q^n))$ when the number of variables goes to infinity is trivial. But using Theorem 3.4.1 we can even say more.

Lemma 4.2.5. *Given $n, m \in \mathbb{N}$ and $1 < p \leqslant 2$ there is $C_p > 0$ such that*

$$\chi_{p,q}((z^\alpha)_{\alpha\in\Lambda(m,n)}) \leqslant C_p^m,$$

where $q = q(p) = (mp')'$.

Proof. Take $P \in \mathcal{P}(^m\mathbb{C}^n)$ and $(\theta_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$. Fix $z \in B_{\ell_q^n}$ by Theorem 3.4.1 we have

$$\left| \sum_{\mathbf{j}\in\mathcal{J}(m,n)} \theta_{\mathbf{j}} c_{\mathbf{j}}(P) z_{\mathbf{j}} \right| \leqslant \sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}| \leqslant m^{d_r} \|P\|_{\mathcal{P}(^m\ell_r^n)}.$$

Taking supremum over $z \in B_{\ell_q^n}$ it follows

$$\left\| \sum_{\mathbf{j}\in\mathcal{J}(m,n)} \theta_{\mathbf{j}} c_{\mathbf{j}}(P) z_{\mathbf{j}} \right\|_{\mathcal{P}(^m\ell_q^n)} \leqslant m^{d_r} \|P\|_{\mathcal{P}(^m\ell_r^n)},$$

thanks to its minimality $\chi_{p,q}((z^\alpha)_{\alpha\in\Lambda(m,n)}) \leqslant m^{d_p}$. Oberve that $\left(m^{d_p}\right)^{1/m} \to 0$ when $m \to \infty$, then there is $C_p > 0$ such that $m^{d_p} \leqslant C_p^m$, and this proves the statement of the lemma. $\square$

Notice that Lemma 4.2.5 gives hypercontractivity for $\chi_{(mp')',p}((z^\alpha)_{\alpha\in\Lambda(m,n)})$. Also observe that the space $\ell^n_{(mp')'}$ depends on m, then we are no able to apply Theorem 4.2.2 to conclude, for example, something about $monH_\infty(B_{\ell_p})$. In Chapter 6 we will concentrate our efforts on finding the largest Banach sequence space independent of m for which hypercontractive inequalities as in Theorem 3.4.1 hold for every $m \in \mathbb{N}$. This will give us Theorem 6.2.3, the key to have a better characterization of $monH_\infty(B_{\ell_r})$ and also $monH_b(\ell_r)$ for $1 < r \leqslant 2$.

4.3 The (p,q)-mixed unconditionality constant

We now present some estimates for the asymptotic behavior of the (p,q)-mixed unconditional constant of $\mathcal{P}(^m\mathbb{C}^n)$. Note that in the case $q = p$ we recover the results from [DDGM01].

Theorem 4.3.1. *For each $m \in \mathbb{N}$ we have*

$$\begin{cases} \chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \sim 1 & \textit{for } (I) \quad : [\frac{1}{p} + \frac{m-1}{2m} \leqslant \frac{1}{q} \wedge \frac{1}{p} \leqslant \frac{1}{2}] \textit{ or} \\ & \qquad\qquad [\frac{m-1}{m} + \frac{1}{mr} < \frac{1}{q} \wedge \frac{1}{2} \leqslant \frac{1}{r}], \\ \chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \sim n^{m(\frac{1}{p}-\frac{1}{q}+\frac{1}{2})-\frac{1}{2}} & \textit{for } (II) \quad : [\frac{1}{p} + \frac{m-1}{2m} \geqslant \frac{1}{q} \wedge \frac{1}{p} \leqslant \frac{1}{2}], \\ \chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \sim n^{(m-1)(1-\frac{1}{q})+\frac{1}{p}-\frac{1}{q}} & \textit{for } (III) : [1 - \frac{1}{m} + \frac{1}{mp} \geqslant \frac{1}{q} \;\wedge\; \frac{1}{2} < \frac{1}{p} < 1]. \end{cases}$$

To prove the theorem we need a lemma relating the (p,q)-mixed unconditional constant with $A^m_{p,r}(n)$ and $B^m_{r,q}(n)$ from Chapter 2.

Lemma 4.3.2. *Let $1 \leqslant q, p \leqslant \infty$, then we have*

$$\chi_{p,q}((z^\alpha)_{\alpha\in\Lambda(m,n)}) \leqslant B^m_{r,q}(n)A^m_{p,r}(n) \quad \textit{for every} \quad 1 \leqslant r \leqslant \infty. \tag{4.7}$$

Proof. Let $P(z) = \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}} z_{\mathbf{j}}$ be an m-homogeneous polynomial in n variables and $(\theta_{\mathbf{j}})_{\mathbf{j}\in\mathcal{J}(m,n)}$ be a sequence of complex numbers of modulus one, then

$$\Big\| \sum_{\mathbf{j}\in\mathcal{J}(m,n)} \theta_{\mathbf{j}} c_{\mathbf{j}} z_{\mathbf{j}} \Big\|_{\mathcal{P}(^m\ell_q^n)} \leqslant B^m_{r,q}(n)\Big(\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}|^r \Big)^{\frac{1}{r}} \leqslant B^m_{r,q}(n)A^m_{q,r}(n)\Big\| \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}} z_{\mathbf{j}} \Big\|_{\mathcal{P}(^m\ell_p^n)},$$

for every $1 \leqslant r \leqslant \infty$. □

Before finally proving Theorem 4.3.1 it will be useful to state that for $2 \leqslant p \leqslant \infty$

$$\ell_{\left(\frac{m-1}{2m}+\frac{1}{p}\right)^{-1}} \subset mon\mathcal{P}(^m\ell_p). \tag{4.8}$$

This holds by (3.16) and using that $\ell_{\left(\frac{m-1}{2m}+\frac{1}{p}\right)^{-1}} \subset \ell_p \cdot \ell_{\left(\frac{m-1}{2m}\right)^{-1},\infty}$.

We will also need the following monotonicity result for the mixed unconditional constant.

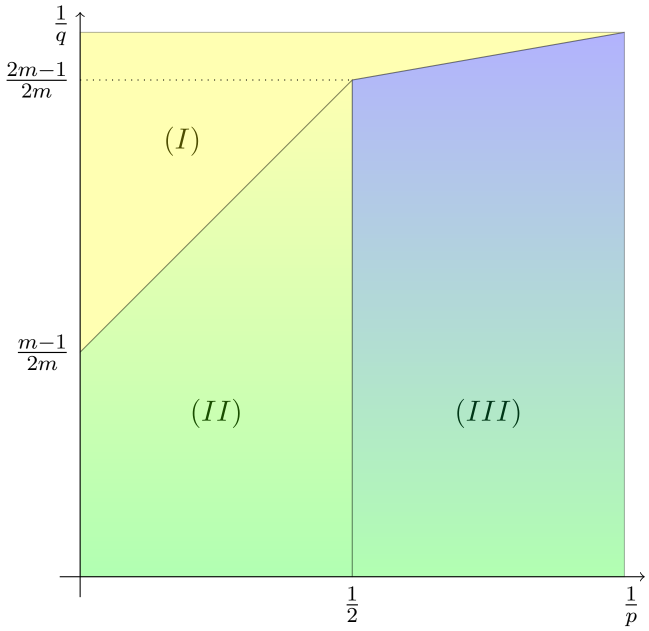

Figure 4.1: Graphical overview of the mixed unconditional constant described in Theorem 4.3.1.

Remark 4.3.3. Given $1 \leqslant r \leqslant q \leqslant \infty$ and $1 \leqslant p \leqslant s \leqslant \infty$

$$\chi_{s,r}\big((z^\alpha)_{\alpha\in\Lambda(m,n)}\big) \leqslant \chi_{p,q}\big((z^\alpha)_{\alpha\in\Lambda(m,n)}\big). \tag{4.9}$$

This is true as, for $P \in \mathcal{P}(^m\mathbb{C}^n)$ and every $(\theta_\alpha)_{\alpha\in\Lambda(m,n)}$ using (2.19) we have

$$\begin{aligned}\|\sum_{\alpha\in\Lambda(m,n)} \theta_\alpha a_\alpha(P) z^\alpha\|_{\mathcal{P}(^m\ell_r)} &\leqslant \|\sum_{\alpha\in\Lambda(m,n)} \theta_\alpha a_\alpha(P) z^\alpha\|_{\mathcal{P}(^m\ell_q)} \\ &\leqslant \chi_{p,q}\left((z^\alpha)_{\alpha\in\Lambda(m,n)}\right)\|P\|_{\mathcal{P}(^m\ell_p^n)} \leqslant \chi_{p,q}\left((z^\alpha)_{\alpha\in\Lambda(m,n)}\right)\|P\|_{\mathcal{P}(^m\ell_s^n)}.\end{aligned}$$

By the minimality of the mixed unconditional constant we have the result.

Proof of Theorem 4.3.1. We will prove the results for $\chi_{p,q}\left((z^\alpha)_{\alpha\in\Lambda(m,n)}\right)$, then by Theorem 4.1.2 will have the same behaviour for $\chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n))$. The proof is divided in cases.

•(I) : Let $p \geqslant 2$. By the inequality in (4.8) we know that $\ell_{q_m} \subset mon(\mathcal{P}(^m\ell_p))$ where

$$\frac{1}{q_m} = \frac{m-1}{2m} + \frac{1}{p},$$

thanks to Theorem 4.2.3 we have

$$\chi_{p,q_m}\left((z^\alpha)_{\alpha\in\Lambda(m,n)}\right) \sim 1.$$

On the other hand, if $p \leqslant 2$, by Lemma 4.2.5 we know that

$$\chi_{p,q_m}\left((z^\alpha)_{\alpha\in\Lambda(m,n)}\right) \sim 1,$$

where $\frac{1}{q_m} = \frac{m-1}{m} + \frac{1}{mp}$.

Therefore, by the monotonicity stated in Remark 4.3.3 it follows $\chi_{p,q}\left((z^\alpha)_{\alpha\in\Lambda(m,n)}\right) \sim 1$ in the region

$$(I): \left[\frac{1}{p} + \frac{m-1}{2m} \leqslant \frac{1}{q} \wedge \frac{1}{p} \leqslant \frac{1}{2}\right] \text{ or } \left[\frac{m-1}{m} + \frac{1}{mp} < \frac{1}{q} \wedge \frac{1}{2} \leqslant \frac{1}{p}\right].$$

•(II) : We know by (I) that $\chi_{p,q_m}(\mathcal{P}(^m\mathbb{C}^n)) \sim 1$, for $\frac{1}{q_m} = \frac{1}{p} + \frac{m-1}{2m}$. We now estimate $\chi_{p,\infty}(\mathcal{P}(^m\mathbb{C}^n))$ for $0 \leqslant \frac{1}{p} \leqslant \frac{1}{2}$. Take $r = 1$. By Proposition 2.2.6 and Theorem 2.2.5 (C) we have

$$B^m_{r,\infty}(n) \sim 1 \ , \ A^m_{p,r}(n) \sim n^{m(\frac{1}{p}+\frac{1}{2})-\frac{1}{2}}.$$

Using Lemma 4.3.2

$$\chi_{p,\infty}(\mathcal{P}(^m\mathbb{C}^n)) \ll n^{m(\frac{1}{p}+\frac{1}{2})-\frac{1}{2}}.$$

Take a polynomial $P \in \mathcal{P}(^m\mathbb{C}^n)$, $P = \sum\limits_{\alpha\in\Lambda(m,n)} a_\alpha z^\alpha$ with $\|P\|_{\mathcal{P}(^m\ell_p^n)} = 1$ and take signs $(\varepsilon_\alpha)_{\alpha\in\Lambda(m,n)}$. Therefore since

$$\Big\|\sum_{\alpha\in\Lambda(m,n)} \varepsilon_\alpha a_\alpha z^\alpha\Big\|_{\mathcal{P}(^m\ell_{q_m}^n)} \leqslant \chi_{p,q_m}(\mathcal{P}(^m\mathbb{C}^n)) \sim 1,$$

$$\Big\|\sum_{\alpha\in\Lambda(m,n)} \varepsilon_\alpha a_\alpha z^\alpha\Big\|_{\mathcal{P}(^m\ell_{\infty}^n)} \leqslant \chi_{p,\infty}(\mathcal{P}(^m\mathbb{C}^n)) \ll n^{m(\frac{1}{p}+\frac{1}{2})-\frac{1}{2}},$$

by (2.23) and using the multilinear interpolation Theorem 1.4.1 for the multilinear form associated to $\sum_{\alpha\in\Lambda(m,n)} \varepsilon_\alpha a_\alpha z^\alpha$, we have that for $\theta \in (0,1)$ and $\frac{1}{q} = \frac{\theta}{q_m} + \frac{1-\theta}{\infty}$,

$$\|\sum_{\alpha\in\Lambda(m,n)} \varepsilon_\alpha a_\alpha z^\alpha\|_{\mathcal{P}(^m\ell_q^n)} \ll n^{(1-\theta)[m(\frac{1}{p}+\frac{1}{2})-\frac{1}{2}]} = n^{m(\frac{1}{p}-\frac{1}{q}+\frac{1}{2})-\frac{1}{2}}.$$

For the lower bound let $P(z) = \sum\limits_{\alpha\in\Lambda(m,n)} \varepsilon_\alpha z^\alpha$ be a unimodular polynomial as in (2.20) with $p \geqslant 2$, then if $w = (\frac{1}{n^{1/q}}, \ldots, \frac{1}{n^{1/q}}) \in S_{\ell_q}$, we have

$$\chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \gg \frac{\|\sum\limits_{\alpha\in\Lambda(m,n)} z^\alpha\|_{\mathcal{P}(^m\ell_q^n)}}{\|\sum\limits_{\alpha\in\Lambda(m,n)} \varepsilon_\alpha z^\alpha\|_{\mathcal{P}(^m\ell_p^n)}} \gg \frac{|\sum\limits_{\alpha\in\Lambda(m,n)} w^\alpha|}{n^{m(\frac{1}{2}-\frac{1}{p})+\frac{1}{2}}} \gg \frac{n^{m(1-\frac{1}{q})}}{n^{m(\frac{1}{2}-\frac{1}{p})+\frac{1}{2}}} = n^{m(\frac{1}{p}-\frac{1}{q}+\frac{1}{2})-\frac{1}{2}}.$$

•(III) : For $[\frac{1}{2} \leqslant \frac{1}{p} \ \wedge \ \frac{1}{q} \leqslant \frac{1}{p}]$ let us take $\frac{1}{r} = 1 - \frac{1}{q}$. Note that $\frac{1}{r} \geqslant 1 - \frac{1}{p}$. Then by Proposition 2.2.6 and Theorem 2.2.5 (D),

$$B^m_{r,q}(n) \sim 1 \ , \ A^m_{p,r}(n) \sim n^{(m-1)(1-\frac{1}{q})+\frac{1}{p}-\frac{1}{q}},$$

and therefore

$$\chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \ll n^{(m-1)(1-\frac{1}{q})+\frac{1}{p}-\frac{1}{q}}.$$

As in (II) we will use The Multilinear Interpolation Theorem: take a polynomial $P = \sum_{\alpha\in\Lambda(m,n)} a_\alpha z^\alpha$ with $\|P\|_{\mathcal{P}(^m\ell_p^n)} = 1$ and take signs $(\varepsilon_\alpha)_{\alpha\in\Lambda(m,n)}$. Therefore since

$$\Big\| \sum_{\alpha\in\Lambda(m,n)} \varepsilon_\alpha a_\alpha z^\alpha \Big\|_{\mathcal{P}(^m\ell_{q_m}^n)} \leqslant \chi_{p,q_m}(\mathcal{P}(^m\mathbb{C}^n)) \sim 1,$$
$$\Big\| \sum_{\alpha\in\Lambda(m,n)} \varepsilon_\alpha a_\alpha z^\alpha \Big\|_{\mathcal{P}(^m\ell_p^n)} \leqslant \chi_{p,p}(\mathcal{P}(^m\mathbb{C}^n)) \sim n^{(m-1)(1-\frac{1}{p})},$$

we have, by (2.23) and Theorem 1.4.1, that for $\theta \in (0,1)$ and $\frac{1}{q} = \frac{\theta}{q_m} + \frac{1-\theta}{p}$,

$$\Big\| \sum_{\alpha\in\Lambda(m,n)} \varepsilon_\alpha a_\alpha z^\alpha \Big\|_{\mathcal{P}(^m\ell_q^n)} \ll n^{(1-\theta)[(m-1)(1-\frac{1}{p})]} = n^{(m-1)(1-\frac{1}{q})+\frac{1}{p}-\frac{1}{q}}.$$

For the lower bound let $P(z) = \sum_{\alpha\in\Lambda(m,n)} \varepsilon_\alpha z^\alpha$ be a unimodular polynomial as in (2.20) with $1 \leqslant p \leqslant 2$, we have

$$\chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n)) \gg \frac{\| \sum_{\alpha\in\Lambda(m,n)} z^\alpha \|_{\mathcal{P}(^m\ell_q^n)}}{\| \sum_{\alpha\in\Lambda(m,n)} \varepsilon_\alpha z^\alpha \|_{\mathcal{P}(^m\ell_p^n)}} \gg \frac{n^{m(1-\frac{1}{q})}}{n^{1-\frac{1}{p}}} = n^{(m-1)(1-\frac{1}{q})+\frac{1}{p}-\frac{1}{q}}. \qquad \square$$

The study we made in Theorem 4.3.1 assumes that the homogeneity degree of the polynomials m is fixed. In particular we do not worry to deeply investigate the dependence of the mixed unconditional constant on m. For example we do not care about hypercontractivity in the bounds we get for this constant. In Chapter 5, Chapter 6 and Chapter 7 we explore objects and aspects of some families of holomorphic functions that will require to look into this deeper comprehension of this constants. Hypercontractivity will be a key property.

Chapter 5

Bohr radius

While working to understand the Dirichlet series, Harald Bohr managed to connect them with holomorphic functions in infinite many variables through what we call the Bohr transform. This cycle of ideas brought Bohr to ask whether it is possible to compare the absolute value of a power series in one complex variable with the sum of the absolute value of its coefficients. He manged to prove the following result nowadays referred as *Bohr's inequality*:

The radius $r = \frac{1}{3}$ is the largest value for which the following inequality holds:

$$\sum_{n\geqslant 0} |a_n| r^n \leqslant \sup_{z\in\mathbb{D}} |\sum_{n\geqslant 0} a_n z^n|, \tag{5.1}$$

for every holomorphic function $f(z) = \sum_{n\geqslant 0} a_n z^n$ bounded on the unit disk $\mathbb{D}$.

As a matter of fact, Bohr's paper [Boh14], compiled by G. H. Hardy from correspondence, indicates that Bohr initially obtained the radius $\frac{1}{6}$, but this was quickly improved to the sharp result by M. Riesz, I. Schur, and N. Wiener, independently. Bohr's article presents both his own proof and the one of his colleagues.

5.1 The n-dimensional Bohr radius

This interesting inequality was overlooked during many years until the end of the twentieth century. In particular, Dineen and Timoney [DT89], Dixon [Dix95], Boas and Khavinson [KB97], Aizenberg [Aiz00], Boas [Boa00] and Bombieri and Bourgain [BB04] retook this work and use it in different contexts and/or generalize it. Several of these authors analyzed if a similar phenomenon occurs for power series in many variables. For each Reinhardt domain $\mathcal{R}$, they introduced the notion of the *Bohr radius* $K(\mathcal{R})$ as the biggest $r \geqslant 0$ such that for every analytic function $f(z) = \sum_\alpha a_\alpha z^\alpha$ bounded on $\mathcal{R}$, it holds:

$$\sup_{z\in r\cdot\mathcal{R}} \sum_\alpha |a_\alpha z^\alpha| \leqslant \sup_{z\in\mathcal{R}} |f(z)|. \tag{5.2}$$

Note that with this notation, Bohr's inequality can be formulated simply as $K(\mathbb{D}) = \frac{1}{3}$. Surprisingly, the exact value of the Bohr radius is unknown for any other domain. The central results of [KB97, Boa00] contained a (partial) successful estimate for the Bohr radius for the complex unit balls of ℓ_p^n, $1 \leqslant p \leqslant \infty$. There the authors reached, for every $1 \leqslant p \leqslant \infty$, the following asymptotic bounds (see [Boa00, Theorem 3]),

$$\frac{1}{3\sqrt[3]{e}} \left(\frac{1}{n}\right)^{1-\frac{1}{\min(p,2)}} \leqslant K(B_{\ell_p^n}) \leqslant 3 \left(\frac{\log(n)}{n}\right)^{1-\frac{1}{\min(p,2)}}. \tag{5.3}$$

The gap between the upper and lower estimates in these papers leaded many efforts to compute the exact asymptotic order of $K(B_{\ell_p^n})$, for $1 \leqslant p \leqslant \infty$.

To obtain the upper bounds Boas [Boa00] generalized a theorem of Kahane-Salem-Zygmund on random trigonometric polynomials [Kah93, Theorem 4 in Chapter 6], which gives (by the use of a probabilistic argument) the existence of homogeneous polynomials with “large coefficients” and “relatively small” uniform norm. This technique was refined by Bayart in [Bay12] from where we extracted (2.20).

The lower bound needed different techniques. In [DGM03] Defant, García and Maestre related the Bohr radius with the study of unconditionality in spaces of homogeneous polynomials via some concepts of the local theory of Banach spaces (see also [DP06]). Although at that moment this did not give optimal asymptotic bounds, it started a way with which $K(B_{\ell_p^n})$ would be obtained.

They were Defant, Frerick, Ortega-Cerdá, Ounaïes and Seip [DFOC$^+$11] who made an incredible contribution in the problem giving the exact asymptotic value of $K(B_{\ell_\infty^n})$. This work, in some sense, marked the path of the whole area over the last years. The authors involved into the game the classical Bohnenblust-Hille inequality, which was used to compute Bohr's convergence width eighty years before. The groundbreaking progress consisted in showing that C_m in Theorem 2.1.2 is in fact *hypercontractive*. They showed C_m can be taken less than or equal to C^m for some absolute constant $C > 0$. With this at hand they proved that $K(B_{\ell_\infty^n})$ behaves asymptotically as $\sqrt{\frac{\log(n)}{n}}$.

In fact much more can be said about $K(B_{\ell_\infty^n})$: Bayart, Pellegrino and Seoane [BPSS14] push these techniques further in an amazingly ingenious way to obtain that

$$\lim_{n\to\infty} \frac{K(B_{\ell_\infty^n})}{\sqrt{\frac{\log(n)}{n}}} = 1.$$

Since $K(B_{\ell_\infty^n})$ bounds from below the radius $K(\mathcal{R})$ for any other Reinhardt domain $\mathcal{R}$, the range where $p \geqslant 2$ easily follows. The solution of the case $p < 2$, required quite different methods. A celebrated theorem proved independently by Pisier [Pis86] and Schütt [Sch78] allows to study unconditional bases in spaces of multilinear forms in terms of some invariants such as the local unconditional structure or the Gordon-Lewis property. These results have their counterpart in the context of spaces of polynomials as shown in [DDGM01], replacing the full tensor product by the symmetric one.

Defant and Frerick [DF11] (continuing their previous work given in [DF06]) established some sort of extension of Pisier-Schütt result to the symmetric tensor product with accurate bounds and gave a new estimate on the Gordon-Lewis constant of the symmetric tensor product. As a byproduct, they found the exact asymptotic growth for the Bohr radius on the unit ball of the spaces ℓ_p^n.

The aforementioned results give the following relation for the Bohr radius.

Theorem 5.1.1. *[DFOC+11, DF11] For* $1 \leqslant p \leqslant \infty$*, we have*

$$K(B_{\ell_p^n}) \sim \left(\frac{\log(n)}{n}\right)^{1-\frac{1}{\min\{p,2\}}}. \tag{5.4}$$

The proof of the exact asymptotic behavior of $K(B_{\ell_p^n})$ given in [DF11] for $p < 2$, as mentioned before, use "sophisticated machinery" from the Banach space theory. Inspired by recent results from the general theory of Dirichlet series, in [BDS19] Bayart, Defant and Schlüters give upper estimates for the unconditional basis constants of spaces of polynomials on ℓ_p spanned by finite sets of monomials, which avoid the use of this "machinery". This perspective gives a new and, in a sense, clearer proof of Theorem 5.1.1 for the case $p < 2$.

5.2 Mixed Bohr radius

In this chapter we aim to continue the study of the Bohr radius for mixed Reinhardt domains. Let $\mathcal{R}$ and $\mathcal{S}$ be two Reinhardt domains in $\mathbb{C}^n$. The mixed Bohr radius $K(\mathcal{R}, \mathcal{S})$ is defined as the biggest number $r \geqslant 0$ such that for every analytic function $f(z) = \sum_\alpha a_\alpha z^\alpha$ bounded on $\mathcal{R}$, it holds:

$$\sup_{z \in r\cdot\mathcal{S}} \sum_\alpha |a_\alpha z^\alpha| \leqslant \sup_{z\in\mathcal{R}} |f(z)|. \tag{5.5}$$

We will focus in the case where $\mathcal{R}$ and $\mathcal{S}$ are the closed unit balls of ℓ_p^n and ℓ_q^n for $1 \leqslant p, q \leqslant \infty$. Note that $K(B_{\ell_p^n})$ using this new notation is just $K(B_{\ell_p^n}, B_{\ell_p^n})$. The following theorem provides the correct asymptotic estimates of $K(B_{\ell_p^n}, B_{\ell_q^n})$ for the full range of p's and q's.

Theorem 5.2.1. *Let* $1 \leqslant p, q \leqslant \infty$*, with* $q \neq 1$*. The asymptotic growth of the* (p,q)*-Bohr radius is given by*

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \sim \begin{cases} 1 & \text{if (I): } 2 \leqslant p \leqslant \infty \ \wedge\ \frac{1}{2} + \frac{1}{p} \leqslant \frac{1}{q}, \\ \frac{\sqrt{\log(n)}}{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}} & \text{if (II): } 2 \leqslant p \leqslant \infty \ \wedge\ \frac{1}{2} + \frac{1}{p} > \frac{1}{q}, \\ \frac{\log(n)^{1-\frac{1}{p}}}{n^{1-\frac{1}{q}}} & \text{if (III): } 1 \leqslant p \leqslant 2. \end{cases}$$

For $q = 1$ *and every* $1 \leqslant p \leqslant \infty$*,* $K(B_{\ell_p^n}, B_{\ell_q^n}) \sim 1$*.*

As for $K(B_{\ell_p^n})$, the upper bounds are obtained using random polynomials with adequate coefficients and relatively small norm [Boa00, DGM04, Bay12]. To obtain the lower bounds the proof is divided in several cases. For $p < 2$ we have combined an appropriate way to

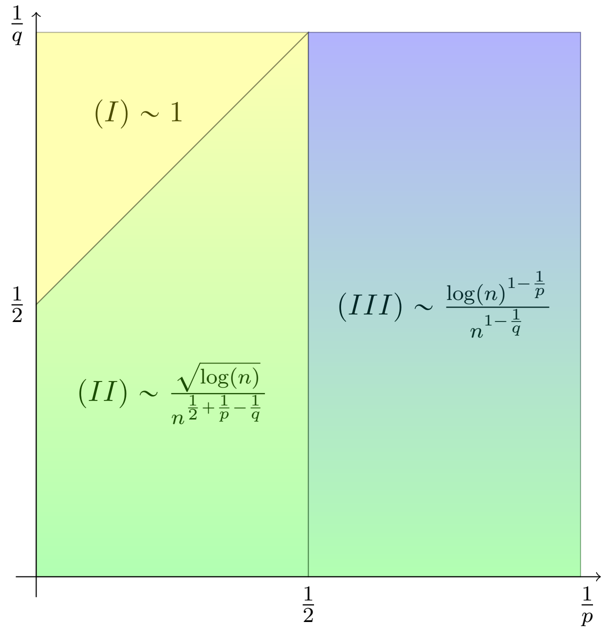

Figure 5.1: Graphical overview of the mixed Bohr radius described in Theorem 5.2.1.

divide and distinguish certain subsets of monomials together with the upper estimates for the unconditional basis constants of spaces of polynomials on ℓ_p spanned by finite sets of monomials given in [BDS19]. The interplay between monomial convergence and mixed unconditionality for spaces of homogeneous polynomials presented in [DMP09, Theorem 5.1.] (which, of course, gives information on the Bohr radius) is crucial for the case $p > 2$. We have strongly used some recent inclusion for the set of monomial convergence $monH_\infty(B_{\ell_p})$ $p \geqslant 2$ given in [DMP09, BDF$^+$17]. Therefore, it is worth noting that the techniques and results developed in the last years were fundamental for our proof.

Following the notation of [BDS19], given a subset $\mathcal{J} \subset \mathcal{J}(m,n)$, we call

$$\mathcal{J}^* = \{\mathbf{j} \in \mathcal{J}(m-1,n) : \text{ there is } k \geqslant 1, (\mathbf{j},k) \in \mathcal{J}\}.$$

5.3 Homogeneous mixed Bohr radius and mixed unconditionality

Recall that $K(B_{\ell_p^n}, B_{\ell_q^n})$ stands for the n-dimensional (p,q)-Bohr radius. That is, $K(B_{\ell_p^n}, B_{\ell_q^n})$ denotes the greatest constant $r > 0$ such that for every entire function $f = \sum_\alpha a_\alpha z^\alpha$ in n-complex variables, we have the following (mixed) Bohr-type inequality

$$\sup_{z \in r \cdot B_{\ell_q^n}} \sum_\alpha |a_\alpha z^\alpha| \leqslant \sup_{z \in B_{\ell_p^n}} |f(z)|.$$

In the same way, the m-homogeneous mixed Bohr radius, $K_m(B_{\ell_p^n}, B_{\ell_q^n})$, is defined as the greatest $r > 0$ such that for every $P(z) = \sum_{\alpha\in\Lambda(m,n)} a_\alpha z^\alpha \in \mathcal{P}(^m\mathbb{C}^n)$ it follows

$$\sup_{z\in B_{\ell_q^n}} \sum_{\alpha\in\Lambda(m,n)} |a_\alpha z^\alpha| r^m = \sup_{z\in B_{\ell_q^n}} |\sum_{\alpha\in\Lambda(m,n)} a_\alpha z^\alpha| \leqslant \|P\|_{\mathcal{P}(^m\ell_p^n)}.$$

It is plain that $K(B_{\ell_p^n}, B_{\ell_q^n}) \leqslant K_m(B_{\ell_p^n}, B_{\ell_q^n})$.

Remark 5.3.1.

$$K_m(B_{\ell_p^n}, B_{\ell_q^n}) = \frac{1}{\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}}.$$

Proof. Given $P \in \mathcal{P}(^m\mathbb{C}^n)$ and for any $(\theta_\alpha)_{\alpha\in\Lambda(m,n)}$ we have

$$\begin{aligned}\| \sum_{\alpha\in\Lambda(m,n)} \theta_\alpha a_\alpha(P) z^\alpha \|_{\mathcal{P}(^m\ell_q^n)} &\leqslant \| \sum_{\alpha\in\Lambda(m,n)} |a_\alpha(P) z^\alpha| \|_{\mathcal{P}(^m\ell_q^n)} \\ &= \| \sum_{\alpha\in\Lambda(m,n)} |a_\alpha(P) z^\alpha| (K_m(B_{\ell_p^n}, B_{\ell_q^n}))^m \|_{\mathcal{P}(^m\ell_q^n)} \frac{1}{(K_m(B_{\ell_p^n}, B_{\ell_q^n}))^m} \\ &\leqslant \frac{1}{(K_m(B_{\ell_p^n}, B_{\ell_q^n}))^m} \|P\|_{\mathcal{P}(^m\ell_p^n)},\end{aligned}$$

which leads to the inequality $\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m} \leqslant \frac{1}{K_m(B_{\ell_p^n}, B_{\ell_q^n})}$. On the other hand, for $P \in \mathcal{P}(^m\mathbb{C}^n)$ take $\theta_\alpha = \frac{\overline{a_\alpha(P)}}{|a_\alpha(P)|}$. Then we have

$$\begin{aligned}\| \sum_{\alpha\in\Lambda(m,n)} |a_\alpha(P) z^\alpha| \|_{\mathcal{P}(^m\ell_q^n)} &= \| \sum_{\alpha\in\Lambda(m,n)} \theta_\alpha a_\alpha(P) z^\alpha \|_{\mathcal{P}(^m\ell_q^n)} \\ &\leqslant \chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n)) \|P\|_{\mathcal{P}(^m\ell_p^n)},\end{aligned}$$

or equivalently,

$$\sup_{z\in B_{\ell_q^n}} \sum_{\alpha\in\Lambda(m,n)} |a_\alpha z^\alpha| \left(\frac{1}{\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}} \right)^m \leqslant \|P\|_{\mathcal{P}(^m\mathbb{C}^n)},$$

which means $\frac{1}{\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}} \leqslant K_m(B_{\ell_p^n}, B_{\ell_q^n})$. □

It will be useful to remember a classic result due to F. Wiener (see [KB97]) which asserts that for every holomorphic function f written as the sum of m-homogeneous polynomials as $f = \sum_{m\geqslant 1} P_m + a_0$ and such that $\sup_{z\in B_{\ell_p^n}} |f(z)| \leqslant 1$ it holds

$$\|P_m\|_{\mathcal{P}(^m\ell_p^n)} \leqslant 1 - |a_0|^2, \tag{5.6}$$

for every $m \in \mathbb{N}$.

In general this inequality is presented for the uniform norm on the polydisk $\|\cdot\|_{\mathcal{P}(^m\ell_\infty^n)}$ (i.e., $p=\infty$), but this version easily follows by a standard modification of the original argument (for $z \in B_{\ell_p^n}$ consider the auxiliary function $g : \mathbb{C}^n \to \mathbb{C}$ given by $g(w) := f(w\cdot z)$).

The next lemma is an adaption of the case $p=q$, see [DGM03, Theorem 2.2.] and constitutes the basic link between Bohr radius and unconditional basis constants of spaces of polynomials on the mixed context (p not necessarily equal to q).

Lemma 5.3.2. *For every $n \in \mathbb{N}$ and $1 \leqslant p,q \leqslant \infty$ it holds*

$$\frac{1}{3}\frac{1}{\sup_{m\geqslant 1}\chi_M(\mathcal{P}(^m\ell_p^n),\mathcal{P}(^m\ell_q^n))^{1/m}} \leqslant K(B_{\ell_p^n},B_{\ell_q^n}) \leqslant \min\left\{\frac{1}{3},\frac{1}{\sup_{m\geqslant 1}\chi_M(\mathcal{P}(^m\ell_p^n),\mathcal{P}(^m\ell_q^n))^{1/m}}\right\}.$$

Proof. From Remark 5.3.1 we have

$$K(B_{\ell_p^n},B_{\ell_q^n}) \leqslant \inf_{m\geqslant 1}\frac{1}{\chi_M(\mathcal{P}(^m\ell_p^n),\mathcal{P}(^m\ell_q^n))^{1/m}} = \frac{1}{\sup_{m\geqslant 1}\chi_M(\mathcal{P}(^m\ell_p^n),\mathcal{P}(^m\ell_q^n))^{1/m}}$$

and due to Bohr's inequality we know $K(\mathbb{D})=\frac{1}{3}$ as it is clear that $K(B_{\ell_p^n},B_{\ell_q^n}) \leqslant K(\mathbb{D})$ for every $n \in \mathbb{N}$ the right hand side inequality holds. For the left hand side inequality let us take some holomorphic function f, without loss of generality let us assume $\sup_{z\in B_{\ell_p^n}}|f(z)|=1$, and consider its decomposition as a sum of m-homogeneous polynomials $f=\sum\limits_{m\geqslant 0}P_m$. For every $m \in \mathbb{N}_0$ it holds $P_m(z)=\sum\limits_{\alpha\in\Lambda(m,n)}a_\alpha(f)z^\alpha$, thus taking $\rho=\sup_{m\geqslant 1}\chi_M(\mathcal{P}(^m\ell_p^n),\mathcal{P}(^m\ell_q^n))^{1/m}$ and using Remark 5.3.1 again it follows

$$\Big\|\sum_{\alpha\in\Lambda(m,n)}|a_\alpha(f)|\left(\frac{z}{\rho}\right)^\alpha\Big\|_{\mathcal{P}(^m\ell_q^n)} \leqslant \Big\|\sum_{\alpha\in\Lambda(m,n)}a_\alpha(f)z^\alpha\Big\|_{\mathcal{P}(^m\ell_p^n)}.$$

Applying the above mentioned Wiener's result for some $w \in B_{\ell_q^n}$ we have that

$$\begin{aligned}\sum_{m\geqslant 0}\sum_{\alpha\in\Lambda(m,n)}|a_\alpha(f)|\left(\frac{w}{3\rho}\right)^\alpha &\leqslant |a_0(f)|+\sum_{m\geqslant 1}\frac{1}{3^m}\Big\|\sum_{\alpha\in\Lambda(m,n)}a_\alpha(f)z^\alpha\Big\|_{\mathcal{P}(^m\ell_p^n)}\\ &\leqslant |a_0(f)|+\sum_{m\geqslant 1}\frac{1}{3^m}(1-|a_0(f)|^2)\\ &\leqslant |a_0(f)|+\frac{1-|a_0(f)|^2}{2}\\ &\leqslant 1=\sup_{z\in B_{\ell_p^n}}|f(z)|,\end{aligned}$$

where last inequality holds as $|a_0(f)| \leqslant \sup_{z\in B_{\ell_p^n}}|f(z)|=1$. The last chain of inequalities and the maximality of mixed Bohr radius lead us to $\frac{1}{3\rho} \leqslant K(B_{\ell_p^n},B_{\ell_q^n})$ as we wanted to prove. □

The previous lemma shows that understanding $K(B_{\ell_p^n}, B_{\ell_q^n})$ translates into seeing how the constant $\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}$ behaves. Unfortunately, the results on the asymptotic growth of the (mixed) unconditional constant $\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))$ (for fixed m) as $n \to \infty$ from Theorem 4.3.1 are not useful here. As it can be seen in Lemma 5.3.2, we need to understand how the value of $\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}$ grows by moving both the number of variables, n, and the degree of homogeneity, m. But beyond this, they give a guideline of what to expect (at least what the different regions in Figure 5.1 should look like).

The heuristic to interpret the different regions in Theorem 5.2.1 is the following: if we assume that the homogeneity degree is very large ($m \to \infty$) in Theorem 4.3.1 then the graph in Figure 4.1 transforms into the one presented in Figure 5.1. All this, together with the upper bounds that one gets after using classical random polynomials (see Section 5.4, somehow the easy part) helped us define where to aim to prove lower bounds.

5.4 Upper bounds

Upper bounds constitute the easy part: we will use the classical probabilistic approach.

We will also need the following remark which is an easy calculus exercise.

Remark 5.4.1. For every positive numbers $a, b > 0$ and $n \in \mathbb{N}$, the function $f : \mathbb{R}_{>0} \to \mathbb{R}$ given by $f(x) := x^a n^{\frac{b}{x}}$ attains its minimum at $x = \log(n)\frac{b}{a}$.

Proof of the upper bounds of Theorem 5.2.1. Upper bounds for the case $\frac{1}{p} + \frac{1}{2} \leqslant \frac{1}{q}$ and $q = 1$ in Theorem 5.2.1 are trivial.

Suppose now $\frac{1}{p} + \frac{1}{2} > \frac{1}{q}$ and let $(\varepsilon_\alpha)_{\alpha \in \Lambda(m,n)} \subset \{-1, 1\}$ signs such that

$$\Big\| \sum_{\alpha \in \Lambda(m,n)} \varepsilon_\alpha \frac{m!}{\alpha!} z^\alpha \Big\|_{B_{\ell_p^n}} \leqslant K_{m,p}\, n^{1-\frac{1}{p}},$$

as in (2.20) where, in this case, $K_{m,p} \leqslant C \log(m)^{1-\frac{1}{p}} m!^{1-\frac{1}{p}}$. Taking $z_0 = (\frac{1}{n^{1/q}}, \ldots, \frac{1}{n^{1/q}}) \in B_{\ell_q}$ we can conclude that

$$\begin{aligned}
n^{m(1-\frac{1}{q})} &= \sum_{\alpha \in \Lambda(m,n)} \frac{m!}{\alpha!} \left(\frac{1}{n^{\frac{1}{q}}}\right)^m \\
&\leqslant \Big\| \sum_{\alpha \in \Lambda(m,n)} |\varepsilon_\alpha| \frac{m!}{\alpha!} z^\alpha \Big\|_{B_{\ell_q^n}} \\
&\leqslant \chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n)) \Big\| \sum_{\alpha \in \Lambda(m,n)} \varepsilon_\alpha \frac{m!}{\alpha!} z^\alpha \Big\|_{B_{\ell_p^n}} \\
&\leqslant \chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))\, K_{m,p}\, n^{1-\frac{1}{p}} \\
&\leqslant \chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))\, C \log(m)^{1-\frac{1}{p}} m!^{1-\frac{1}{p}}\, n^{1-\frac{1}{p}}.
\end{aligned}$$

For $p < 2$ we have, by Stirling's formula (2.16), that

$$\begin{aligned}\frac{1}{\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}} &\leqslant \left(Cn^{\frac{1}{p'}} (\log(m)m!)^{\frac{1}{p'}}\right)^{1/m} \frac{1}{n^{\frac{1}{q'}}} \\ &\leqslant C\frac{1}{n^{\frac{1}{q'}}} m^{\frac{1}{p'}} n^{\frac{1}{p'm}}.\end{aligned}$$

Thanks to Lemma 5.3.1, Remark 5.4.1 and the previous inequality

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \leqslant C\frac{1}{n^{\frac{1}{q'}}} \inf_{m\geqslant 1} m^{\frac{1}{p'}} n^{\frac{1}{p'm}} \leqslant C\frac{\log(n)^{\frac{1}{p'}}}{n^{\frac{1}{q'}}}.$$

On the other hand, for $p \geqslant 2$ and $\frac{1}{p} + \frac{1}{2} > \frac{1}{q}$ it follows

$$\begin{aligned}\frac{1}{\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n)))^{1/m}} &\leqslant \left(Cn^{\frac{1}{2}} (\log(m)m!)^{\frac{1}{2}}\right)^{1/m} \frac{1}{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}} \\ &\leqslant C\frac{1}{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}} m^{\frac{1}{2}} n^{\frac{1}{2m}}.\end{aligned}$$

Thus minimizing $m^{\frac{1}{2}} n^{\frac{1}{2m}}$ as in the previous case we get,

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \leqslant C\frac{\sqrt{\log(n)}}{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}},$$

as we wanted to prove. □

5.5 Lower bounds

For the proof of the lower bounds we need to consider four different cases. We begin with the case $q = 1$ and the case $p \leqslant q$, which are the easy ones. Then we study the case $1 < q \leqslant p \leqslant 2$ where we use tools from unconditionality and finally the case $p \geqslant 2$ where the key tool is monomial convergence.

5.5.1 The case $q = 1$

By [Aiz00], $K(B_{\ell_1^n}) \sim 1$. Thus, for any $f(z) = \sum_\alpha a_\alpha z^\alpha$, it follows that

$$\sup_{z\in K(B_{\ell_1^n})\cdot B_{\ell_1^n}} \sum_\alpha |a_\alpha z^\alpha| \leqslant \sup_{z\in B_{\ell_1^n}} |f(z)| \leqslant \sup_{z\in B_{\ell_p^n}} |f(z)|,$$

which implies that $K(B_{\ell_p^n}, B_{\ell_1^n}) \geqslant K(B_{\ell_1^n}) \sim 1$.

5.5.2 The case $p \leqslant q$

For this case we will strongly use Theorem 5.1.1. The case $p \leqslant q$ is an easy corollary of this result. For any $P(z) = \sum_{\alpha \in \Lambda(m,n)} a_\alpha z^\alpha$, it follows that

$$\Big\| \sum_{\alpha \in \Lambda(m,n)} |a_\alpha| z^\alpha \Big\|_{\mathcal{P}(^m \ell_q^n)} \leqslant n^{m(\frac{1}{p}-\frac{1}{q})} \Big\| \sum_{\alpha \in \Lambda(m,n)} |a_\alpha| z^\alpha \Big\|_{\mathcal{P}(^m \ell_p^n)}$$
$$\leqslant n^{m(\frac{1}{p}-\frac{1}{q})} K(B_{\ell_p^n})^{-m} \Big\| \sum_{\alpha \in \Lambda(m,n)} a_\alpha z^\alpha \Big\|_{\mathcal{P}(^m \ell_p^n)},$$

which implies that $K_m(B_{\ell_p^n}, B_{\ell_q^n}) \geqslant K(B_{\ell_p^n}) n^{\frac{1}{q}-\frac{1}{p}}$ for every $m \in \mathbb{N}$. Using Lemma 5.3.2 and Theorem 5.1.1 we have, for $p \leqslant 2$,

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \geqslant \frac{1}{3} n^{\frac{1}{q}-\frac{1}{p}} K(B_{\ell_p^n}) \sim n^{\frac{1}{q}-\frac{1}{p}} \left(\frac{\log(n)}{n} \right)^{1-\frac{1}{p}} = \frac{\log(n)^{1-1/p}}{n^{1-1/q}},$$

and, for $p \geqslant 2$,

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \geqslant \frac{1}{3} n^{\frac{1}{q}-\frac{1}{p}} K(B_{\ell_p^n}) \sim n^{\frac{1}{q}-\frac{1}{p}} \left(\frac{\log(n)}{n} \right)^{1-\frac{1}{2}} = \frac{\sqrt{\log(n)}}{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}}.$$

This concludes the proof.

5.5.3 The bounded decomposition. The case $1 < q < p \leqslant 2$.

To prove the lower bound is correct in this range of values for p and q we introduce the first monomial decomposition of the thesis, the bounded decomposition. In this case we break the whole set of monomials $(z_{\mathbf{j}})_{\mathbf{j} \in \mathcal{J}(m,n)}$ into subsets depending on the maximum degree of its variables. This partition allows us to manage some technical bounds in a subtler way.

The next lemma is the first step we need to prove the lower bound on Theorem 5.2.1 in this case. A fundamental piece on its proof will be the *BDS inequality*.

Lemma 5.5.1. *Let $1 \leqslant q < p \leqslant 2$. Then we have*

$$\chi_M(\mathcal{P}(^m \ell_p^n), \mathcal{P}(^m \ell_q^n)) \leqslant m e^{1+\frac{m-1}{p}} \left(\sum_{\mathbf{j} \in \mathcal{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'}.$$

Proof. Fix $P \in \mathcal{P}(^m \ell_p^n)$ and $u \in \ell_q^n$. Then, by the *BDS inequality* in (2.9), for any $\mathbf{j} \in \mathcal{J}(m,n)^*$,

$$\left(\sum_{k:\ (\mathbf{j},k) \in \mathcal{J}(m,n)} |c_{(\mathbf{j},k)}(P)|^{p'} \right)^{1/p'} = \left(\sum_{k=j_{m-1}}^{n} |c_{(\mathbf{j},k)}(P)|^{p'} \right)^{1/p'} \leqslant m e^{1+\frac{m-1}{p}} |\mathbf{j}|^{1/p} \|P\|_{\mathcal{P}(^m \ell_p^n)}.$$

Now applying the above inequality, Hölder's inequality (two times) and the multinomial formula we have

$$\begin{aligned}
\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P)||u_{\mathbf{j}}| &= \sum_{\mathbf{j}\in\mathcal{J}(m,n)^*} \left(\sum_{k\colon (\mathbf{j},k)\in\mathcal{J}(m,n)} |c_{(\mathbf{j},k)}||u_{\mathbf{j}}||u_k| \right) \\
&\leqslant \sum_{\mathbf{j}\in\mathcal{J}(m,n)^*} |u_{\mathbf{j}}| \left(\sum_{k\colon (\mathbf{j},k)\in\mathcal{J}(m,n)} |c_{(\mathbf{j},k)}|^{q'} \right)^{1/q'} \left(\sum_k |u_k|^q \right)^{1/q} \\
&\leqslant \sum_{\mathbf{j}\in\mathcal{J}(m,n)^*} |u_{\mathbf{j}}| \left(\sum_{k\colon (\mathbf{j},k)\in\mathcal{J}(m,n)} |c_{(\mathbf{j},k)}|^{p'} \right)^{1/p'} \|u\|_q \\
&\leqslant me^{1+\frac{m-1}{p}} \sum_{\mathbf{j}\in\mathcal{J}(m,n)^*} |\mathbf{j}|^{1/p}|u_{\mathbf{j}}|\|u\|_q\|P\|_{\mathcal{P}(^m\ell_p^n)} \\
&\leqslant me^{1+\frac{m-1}{p}} \left(\sum_{\mathbf{j}\in\mathcal{J}(m,n)^*} |\mathbf{j}||u_{\mathbf{j}}|^q \right)^{1/q} \left(\sum_{\mathbf{j}\in\mathcal{J}(m,n)^*} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \|u\|_q\|P\|_{\mathcal{P}(^m\ell_p^n)} \\
&\leqslant me^{1+\frac{m-1}{p}} \left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,n)} |\mathbf{j}||u_{\mathbf{j}}|^q \right)^{1/q} \left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \|u\|_q\|P\|_{\mathcal{P}(^m\ell_p^n)} \\
&= me^{1+\frac{m-1}{p}} \left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \|u\|_q^m\|P\|_{\mathcal{P}(^m\ell_p^n)},
\end{aligned}$$

which gives the desired inequality. □

The key to prove the lower bound is to obtain good bounds for the sum on the right hand side of the previous lemma.

Lemma 5.5.2. *Let $1 < q \leqslant p \leqslant 2$, for large enough n it follows*

$$\left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \leqslant C^m \frac{n^{m/q'}}{\log(n)^{m/p'}},$$

for every $m \in \mathbb{N}$.

With this at hand it is easy to prove the remaining lower bounds of the case $p \leqslant 2$ of the main theorem.

Proof of the lower bound of the case $1 < q < p \leqslant 2$ on Theorem 5.2.1. Thanks to Lemma 5.3.2 it is enough to prove that

$$\frac{\log(n)^{1-1/p}}{n^{1-1/q}} \ll \frac{1}{\sup_{m\geqslant 1} \chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}} = \inf_{m\geqslant 1} \frac{1}{\chi_M(\mathcal{P}(^m\ell_p^n), \mathcal{P}(^m\ell_q^n))^{1/m}}, \quad (5.7)$$

which follows by Lemma 5.5.1 and Lemma 5.5.2. □

The proof of Lemma 5.5.2 will require some hard work and the insight given by an specific monomial decomposition, the *bounded decomposition*. The idea of this decomposition is to split the set of monomials on those which have the degrees of all their variables bounded (in some sense) and those that don't.

The bounded monomial decomposition: We define for any $1 \leqslant k \leqslant m$ the k-*bounded index set* as

$$\Lambda_k(m,n) = \{\alpha \in \Lambda(m,n) : \alpha_i \leqslant k \text{ for all } 1 \leqslant i \leqslant n\}.$$

Recall that F is the bijective mapping connecting $\Lambda(m,n)$ and $\mathcal{J}(m,n)$, we denote

$$\mathcal{J}_k(m,n) = F^{-1}(\Lambda_k(m,n)),$$

for the corresponding k-bounded index set seen in $\mathcal{J}(m,n)$. Observe that for any $1 \leqslant k \leqslant m$ and $\mathbf{j} \in \mathcal{J}_k(m,n)$ the following inequalities hold:

$$|\mathbf{j}| \geqslant \frac{m!}{k!^{\lceil \frac{m}{k} \rceil}} \geqslant \frac{m!}{k!^{\frac{m}{k}+1}} \geqslant C^m \frac{m^m}{k^{m+k}}, \tag{5.8}$$

$$\left(\sum_{\mathbf{j} \in \mathcal{J}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \leqslant m^{1/q'} \max_{k=1,\ldots,m-1} \left\{ \left(\sum_{\mathbf{j} \in \mathcal{J}_k(m-1,n) \cap \mathcal{J}^c_{k-1}(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} \right\}. \tag{5.9}$$

Finally,

$$\begin{aligned} |\mathcal{J}^c_{k-1}(m-1,n)| &\leqslant n|\mathcal{J}(m-k-1,n)| \\ &\leqslant n\binom{n+m-k-2}{m-k-1} \leqslant n\frac{(n+m-k-2)^{m-k-1}}{(m-k-1)!}, \end{aligned} \tag{5.10}$$

since $\mathbf{j} \in \mathcal{J}^c_{k-1}(m-1,n)$ requires that at least one of the variables is at the power of k (note that this bound is excessive, given that there are many repetitions at the time of counting, but it will be adequate for our purposes). For the particular case $m \leqslant n$ we can extract from inequality (5.10) the fact that

$$|\mathcal{J}^c_{k-1}(m-1,n)| \leqslant 2^m \frac{n^{m-k}}{(m-k-1)!}. \tag{5.11}$$

We are now ready to prove Lemma 5.5.2.

Proof of Lemma 5.5.2. Suppose that n is large enough in order to have

$$\log(n)^c \leqslant n, \tag{5.12}$$

where $c > 0$ is a constant that will be specified later.

We will split the proof in two cases.

• (i) $m \leqslant \log(n)^{\frac{q'}{p'}}$.

Note that if we take in (5.12) $c \geqslant \frac{q'}{p'}$, then $m \leqslant n$. Thus, by (5.8) and (5.11), we have for each $1 \leqslant k \leqslant m-1$,

$$\begin{aligned}\left(\sum_{\mathbf{j} \in \mathcal{J}_k(m-1,n) \cap \mathcal{J}_{k-1}^c(m-1,n)} |\mathbf{j}|^{(1/p-1/q)q'} \right)^{1/q'} &\leqslant C^m |\mathcal{J}_{k-1}^c(m-1,n)|^{1/q'} \frac{k^{(m+k)(\frac{1}{q}-\frac{1}{p})}}{m^{m(\frac{1}{q}-\frac{1}{p})}} \\ &\leqslant C^m \left(\frac{n^{m-k}}{(m-k-1)!} \right)^{\frac{1}{q'}} \frac{k^{(m+k)(\frac{1}{q}-\frac{1}{p})}}{m^{m(\frac{1}{q}-\frac{1}{p})}}.\end{aligned}$$

Thus, by (5.9), we will prove the lemma if we are able to show that this last expression is $\leqslant C^m \frac{n^{m/q'}}{\log(n)^{m/p'}}$, for some constant $C > 0$. Therefore, it suffices to prove that, if $\beta := (\frac{1}{q} - \frac{1}{p})q'$,

$$\frac{k^{\beta(m+k)}}{(m-k-1)! m^{\beta m}} \leqslant C^m \frac{n^k}{\log(n)^{m(\beta+1)}}. \tag{5.13}$$

Let us first suppose that $k \geqslant \min\{m/2, \frac{m}{3\beta}\} = dm$ for some $0 < d < 1$. Note that the left hand side is less than or equal to $m^{\beta m}$, which is $\leqslant \frac{n^{dm}}{\log(n)^{m(\beta+1)}}$ if we choose $cd > \beta(\frac{q'}{p'}+1)+1$ in (5.12) because $m \leqslant \log(n)^{\frac{q'}{p'}}$. Thus we have (5.13) for $k \geqslant dm$.

For $k \leqslant dm$, (5.13) is equivalent to

$$\frac{k^{\beta(m+k)}}{m^{m-k} m^{\beta m}} \leqslant C^m \frac{n^k}{\log(n)^{m(\beta+1)}}, \tag{5.14}$$

for some constant C. If $k = 1$ (5.14) is trivially satisfied. Let $1 < k \leqslant dm$. Using elementary calculus, it can be seen that $k^{\beta k} m^k / k^m \leqslant k^{m/3} m^k / k^m = m^k / k^{2m/3} \ll 1$ for $k \leqslant m/2$. Thus, it is enough to show that

$$\frac{k^{(\beta+1)m}}{m^{(\beta+1)m}} \leqslant C^m \frac{n^k}{\log(n)^{m(\beta+1)}}.$$

Taking logarithms, we see that it suffices to prove that, for some constant $C > 0$ we have,

$$f(\frac{k\log(n)}{m}) - \frac{k\log(n)}{m} \leqslant C, \tag{5.15}$$

where $f(t) = (\beta+1)\log(t)$. Note that there is some $t_0 = t_0(\beta) \geqslant 1$ such that $f(t) \leqslant t^{1/2}$ for every $t \geqslant t_0$. If $\frac{k\log(n)}{m} \geqslant t_0$ then $f(\frac{k\log(n)}{m}) - \frac{k\log(n)}{m} \leqslant \sqrt{\frac{k\log(n)}{m}} - \frac{k\log(n)}{m} \leqslant 0$, and (5.15) is satisfied. On the other hand, if $\frac{k\log(n)}{m} \leqslant t_0$ then (5.15) is fulfilled taking $C = f(t_0)$.

• (ii) For $m \geqslant \log(n)^{\frac{q'}{p'}}$ we just bound $|\mathbf{j}|^{(1/p-1/q)q'}$ by 1, thus we have by Stirling formula,

$$\begin{aligned}\left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,n)}|\mathbf{j}|^{(1/p-1/q)q'}\right)^{1/q'} &\leqslant |\mathcal{J}(m-1,n)|^{1/q'}\\ &= \left(\frac{(n+m-2)!}{(m-1)!(n-1)!}\right)^{1/q'}\\ &\leqslant C^m\left(\left(1+\frac{n}{m-1}\right)^{m-1}\right)^{1/q'}\\ &\leqslant C^m\left(1+\frac{n}{\log(n)^{\frac{q'}{p'}}}\right)^{\frac{m-1}{q'}}\\ &\leqslant C^m\frac{n^{\frac{m}{q'}}}{\log(n)^{\frac{m}{p'}}},\end{aligned}$$

which concludes the proof. □

5.5.4 The case $p \geqslant 2$

For the remaining cases it will be crucial the monomial convergence point of view and its connection with the mixed unconditionality.

Proof of the case $\frac{1}{2}+\frac{1}{p}\leqslant\frac{1}{q}$ on Theorem 5.2.1. By equation (3.9) if $\frac{1}{q}\geqslant\frac{1}{r}=\frac{1}{p}+\frac{1}{2}$ and $p\geqslant 2$ it follows

$$\ell_r\cap B_{\ell_p}\subset monH_\infty(B_{\ell_p}).$$

Since $q\leqslant r\leqslant p$, then $B_{\ell_q}\subset B_{\ell_r}\subset B_{\ell_p}$ and then $B_{\ell_q}\subset\ell_r\cap B_{\ell_p}\subset monH_\infty(B_{\ell_p})$. Finally by Theorem 4.2.2 we have that there is some constant $C=C(p,q)>0$ such that for every $n\in\mathbb{N}$ and p,q fulfilling the previous conditions it holds

$$\frac{1}{\sup\limits_{m\geqslant 1}\left(\chi_M(\mathcal{P}(^m\ell_p^n),\mathcal{P}(^m\ell_q^n))\right)^{1/m}}\geqslant C.$$

As $K(B_{\ell_p^n},B_{\ell_q^n})\leqslant 1$ for every $1\leqslant p,q\leqslant\infty$, the previous inequality and Lemma 5.3.2 lead us to the assertion that for $\frac{1}{q}\geqslant\frac{1}{p}+\frac{1}{2}$

$$K(B_{\ell_p^n},B_{\ell_q^n})\sim 1. \tag{5.16}$$

□

Recall from Theorem 3.2.3 that

$$B = \left\{ z \in \ell_\infty : \limsup_{n\to\infty} \frac{1}{\sqrt{\log(n)}} \left(\sum_{j=1}^{n} |z_j^*|^2 \right)^{1/2} < 1 \right\} \subset monH_\infty(B_{\ell_\infty}).$$

Consider now the Banach space

$$X_\infty = \left\{ z \in \ell_\infty : \sup_{n\geqslant 2} \frac{1}{\sqrt{\log(n)}} \left(\sum_{j=1}^{n} |z_j^*|^2 \right)^{1/2} < \infty \right\}, \tag{5.17}$$

endowed with the norm $\|z\|_{X_\infty} = \sup_{n\geqslant 2} \frac{1}{\sqrt{\log(n)}} \left(\sum_{j=1}^{n} |z_j^*|^2 \right)^{1/2}$. It is not difficult to see that this is Banach sequence space. Observe also that

$$B_{X_\infty} \subset B \subset monH_\infty(B_{\ell_\infty}). \tag{5.18}$$

By Theorem 4.2.2 and expression (5.18) we have for some $C = C(p) > 0$

$$\sup_{n\geqslant 2} \chi_M(\mathcal{P}(^m\ell_\infty^n), \mathcal{P}(^m(X_\infty)_n)) \leqslant C^m. \tag{5.19}$$

Note that the norm in $(X_\infty)_n$ is given by

$$\|(z_1,\ldots,z_n)\|_{(X_\infty)_n} = \sup_{2\leqslant k\leqslant n} \frac{1}{\sqrt{\log(k)}} \left(\sum_{j=1}^{k} |z_j^*|^2 \right)^{1/2}.$$

To complete the study of the mixed Bohr radius for $p \geqslant 2$ it remains to understand the case $\frac{1}{q} < \frac{1}{2} + \frac{1}{p}$.

Proof of the case $\frac{1}{q} < \frac{1}{2} + \frac{1}{p}$ and $p \geqslant 2$ on Theorem 5.2.1. Fix $m \in \mathbb{N}$ and take $P \in \mathcal{P}(^m\mathbb{C}^n)$, $P(z) = \sum_{\alpha\in\Lambda(m,n)} a_\alpha z^\alpha$. By Lemma 5.3.2, it suffices to show that there exists some $C > 0$ such that for every $z \in B_{\ell_q^m}$ it holds

$$\sum_{\alpha\in\Lambda(m,n)} |a_\alpha z^\alpha| \leqslant C^m \left(\frac{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}}{\sqrt{\log(n)}} \right)^m \|P\|_{\mathcal{P}(^m\ell_p^n)}.$$

Consider now $y = (z_1^{\frac{p}{p+2}}, \ldots, z_n^{\frac{p}{p+2}})$ and $w = (z_1^{\frac{2}{p+2}}, \ldots, z_n^{\frac{2}{p+2}})$. It is easy to see that $z = y \cdot w = (y_1 w_1, \ldots, y_n w_n)$, and thus, by (5.19) and Remark 3.1.5, we have

$$\begin{aligned} \sum_{\alpha\in\Lambda(m,n)} |a_\alpha z^\alpha| &= \sum_{\alpha\in\Lambda(m,n)} |a_\alpha w^\alpha y^\alpha| \leqslant C^m \|y\|_{(X_\infty)_n}^m \|P_w\|_{\mathcal{P}(^m\ell_\infty^n)} \\ &\leqslant C^m \|y\|_{(X_\infty)_n}^m \|w\|_{\ell_p^n}^m \|P\|_{\mathcal{P}(^m\ell_p^n)}. \end{aligned}$$

It remains to check that

$$\|y\|_{(X_\infty)_n}\|w\|_{\ell_p^n} \leqslant C\frac{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}}{\sqrt{\log(n)}}.$$

To start, let $1 \leqslant k \leqslant n$ then

$$\begin{aligned}
\|(y_1^*,\dots,y_k^*)\|_{\ell_2^k} &= \|(z_1^*,\dots,z_k^*)\|_{\ell^k_{\frac{2p}{p+2}}}^{\frac{p}{p+2}} \\
&\leqslant \left(\|(z_1^*,\dots,z_k^*)\|_{\ell_q^k} k^{\frac{1}{p}+\frac{1}{2}-\frac{1}{q}}\right)^{\frac{p}{p+2}} \\
&\leqslant \|z\|_{\ell_q^n}^{\frac{p}{p+2}} \left(k^{\frac{1}{p}+\frac{1}{2}-\frac{1}{q}}\right)^{\frac{p}{p+2}},
\end{aligned}$$

so we have

$$\begin{aligned}
\|y\|_{(X_\infty)_n} &= \sup_{2\leqslant k\leqslant n} \frac{1}{\sqrt{\log(k)}}\|(y_1^*,\dots,y_k^*)\|_{\ell_2^k} \\
&\leqslant \sup_{2\leqslant k\leqslant n} \frac{1}{\sqrt{\log(k)}}\|z\|_{\ell_q^n}^{\frac{p}{p+2}} \left(k^{\frac{1}{p}+\frac{1}{2}-\frac{1}{q}}\right)^{\frac{p}{p+2}} \\
&\leqslant C\|z\|_{\ell_q^n}^{\frac{p}{p+2}} \frac{n^{\frac{1}{2}-\frac{1}{q}\frac{p}{p+2}}}{\sqrt{\log(n)}}.
\end{aligned}$$

On the other hand,

$$\|w\|_{\ell_p^n} = \|z\|_{\ell^n_{\frac{2p}{p+2}}}^{\frac{2}{2+p}} \leqslant \|z\|_{\ell_q^n}^{\frac{2}{2+p}} n^{\left(\frac{1}{2}+\frac{1}{p}-\frac{1}{q}\right)\frac{2}{2+p}} = \|z\|_{\ell_q^n}^{\frac{2}{2+p}} n^{\frac{1}{p}-\frac{1}{q}\frac{2}{2+p}}.$$

Finally,

$$\begin{aligned}
\|y\|_{(X_\infty)_n}\|w\|_{\ell_p^n} &\leqslant C\|z\|_{\ell_q^n}^{\frac{p}{p+2}} \frac{1}{\sqrt{\log(n)}} n^{\frac{1}{2}-\frac{1}{q}\frac{p}{p+2}} \|z\|_{\ell_q^n}^{\frac{2}{2+p}} n^{\frac{1}{p}-\frac{1}{q}\frac{2}{2+p}} \\
&= C\|z\|_{\ell_q^n} \frac{n^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}}{\sqrt{\log(n)}},
\end{aligned}$$

as we needed. □

Chapter 6

Monomial convergence for $H_b(\ell_r)$

In this chapter we return to the study of monomial convergence. In particular we focus on the sets of monomial convergence for $H_b(\ell_r)$. In Theorem 6.2.1 we provide a complete characterization of the set of monomial convergence of the space of holomorphic functions of bounded type for $1 < r \leqslant 2$. In Section 6.3 we do so for $monH_b(\ell_{r,s})$ and give very tight lower and upper bounds for $monH_b(\ell_{r,s})$ in particular with $r = s$.

The main tool developed is a novel decomposition of the multi-indices, which allows us to construct any multi-index from two very particular classes (namely, the thetahedral and the even ones). A proper treatment for each of these classes provides bounds for the sum of all the monomials that allow us to prove hypercontractive behaviour for the mixed unconditionality constant between the adequate spaces. This technique is quite different from the usual one, which involves the partition of the multi-index set into suitable subsets, as we did in Section 5.5.3 (see also, for example [DFOC$^+$11, BDS19, BDF$^+$17, GMMa, OCOS09]). The difference now is that one studies two subclasses of multi-indices which decompose all of them (in the fashion of the fundamental theorem of arithmetic, to make an analogy) and manages to conclude something about the entire sum.

6.1 The factorization decomposition

Now we present the second monomial decomposition: *the factorization decomposition.* This decomposition will be the key tool for the lower inclusions in the whole chapter. It will be needed later in the sequel.

Let us be more precise and introduce some notation. A multi-index α is *tetrahedral* if all its entries are either 0 or 1. We consider the set of tetrahedral multi-indices

$$\Lambda_T(m,n) = \big\{\alpha \in \Lambda(m,n) : \alpha_i \in \{0,1\}\big\}.$$

Notice that the set of *tetrahedral multi-indices* is exactly the set of 1-bounded index we introduced in Section 5.5.3, i.e., $\Lambda_T(m,n) = \Lambda_1(m,n)$.

A multi-index is called *even* if all its non-zero entries are even (note that this forces the multi-index to have even order). We consider then the set

$$\Lambda_E(m,n) = \big\{\alpha \in \Lambda(m,n) : \alpha_i \text{ is even for every } i = 1,\dots,n\big\}.$$

Observe that for every $\alpha \in \Lambda_E(m,n)$ there is a unique $\beta \in \Lambda(m/2,n)$ such that $\alpha = 2\beta$.

Remark 6.1.1. Given $\alpha \in \Lambda(M,N)$ define α_T (the tetrahedral part) and α_E (the even part) as

$$\left(\alpha_T\right)_i = \begin{cases} 1 & \text{if } \alpha_i \text{ is odd} \\ 0 & \text{if } \alpha_i \text{ is even} \end{cases} \quad \text{and} \quad \left(\alpha_E\right)_i = \begin{cases} \alpha_i - 1 & \text{if } \alpha_i \text{ is odd} \\ \alpha_i & \text{if } \alpha_i \text{ is even} \end{cases}.$$

If $0 \leqslant k \leqslant M$ is the number of odd entries in α, then clearly $\alpha_T \in \Lambda_T(k,N)$ and $\alpha_E \in \Lambda_E(M-k,N)$ and $\alpha = \alpha_T + \alpha_E$. As $(\alpha_E)_i \leqslant \alpha_i$ for every i then $\alpha_E! \leqslant \alpha!$. On the other hand, $\alpha_T! = 1$, then $\alpha_T!\alpha_E! \leqslant \alpha!$, and

$$|[\alpha]| = \frac{M!}{\alpha!} \leqslant \frac{M!}{\alpha_T!\alpha_E!} = \frac{M!}{(M-k)!k!}\frac{k!}{\alpha_T!}\frac{(M-k)!}{\alpha_E!} = \binom{M}{k}|[\alpha_T]||[\alpha_E]| \leqslant 2^M|[\alpha_T]||[\alpha_E]|.$$

6.2 The case $1 < r \leqslant 2$.

We can now describe the set of monomial convergence of $H_b(\ell_r)$ for $1 < r \leqslant 2$. It happens to be a *Marcinkiewicz space* m_{Ψ_r} where the symbol is given by

$$\Psi_r(n) := \log(n+1)^{1-\frac{1}{r}}, \tag{6.1}$$

for $n \in \mathbb{N}_0$.

Theorem 6.2.1. *For* $1 < r \leqslant 2$*,*

$$monH_b(\ell_r) = m_{\Psi_r} = \left\{ z \in \mathbb{C}^{\mathbb{N}} : \sup_{n \geqslant 1} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-\frac{1}{r}}} < \infty \right\}.$$

We handle the upper and the lower inclusions separately in the following two sections.

6.2.1 The upper inclusion $monH_b(\ell_r) \subset m_{\Psi_r}$

Typically, the way to prove upper inclusions for a set of monomial convergence goes through providing polynomials satisfying certain convenient properties. Over the last years probabilistic techniques have shown to be extremely helpful to find such polynomials. This is, for instance, what is done in [BDF$^+$17, Theorem 2.2], where the probabilistic device is the well known Kahane-Salem-Zygmund inequality. Here we follow essentially the same lines, but using the polynomials provided by Bayart given in 2.20. These polynomials are the main tool for the proof of the upper inclusion. We also need the following result, an extension of [DMP09, Lemma 4.1] whose proof follows the same lines.

Lemma 6.2.2. *Let $\mathcal{R}$ be a Reinhardt domain in a Banach sequence space X and let $(\mathcal{F},(q_n)_n)$ be a Fréchet space of holomorphic functions continuously included in $H_b(\mathcal{R})$. Then, for each $z \in mon(\mathcal{F})$, there exist $C > 0$ and n such that*

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha z^\alpha| \leqslant C q_n(f).$$

for every $f \in \mathcal{F}$. In particular, if $z \in monH_b(X)$, there exists $C > 0$, such that

$$\sum_{\alpha\in\Lambda(m,n)} |c_\alpha(P)z^\alpha| \leqslant C^m \|P\|_{\mathcal{P}(^m X)},$$

for every $P \in \mathcal{P}(^m X)$.

Proof. Given $z \in \mathcal{R}$ meeting (3.6) it clearly holds $\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |a_\alpha(f)z^\alpha| < \infty$ for every $f \in \mathcal{F}(\mathcal{R})$. For the other implication consider the linear mapping

$$\begin{aligned}\Phi_z : \mathcal{F}(\mathcal{R}) &\to \ell_1\left(\mathbb{N}_0^{(\mathbb{N})}\right) \\ f &\mapsto (a_\alpha(f)z^\alpha)_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}},\end{aligned}$$

which is well define as $z \in mon\mathcal{F}(\mathcal{R})$. For $f_n \in \mathcal{F}(\mathcal{R})$ such that $f_n \to f \in \mathcal{F}(\mathcal{R})$ and $\Phi_z(f_n) \to b = (b_\alpha)_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}}$ we have, for any $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$, that $a_\alpha(f_n) \to b_\alpha$. It is easy to see Remark 3.1.3 also holds for $H_b(X)$ with the same proof, using it we have $a_\alpha(f_n) \to a_\alpha(f)$, and due to the uniqueness of the limit $b = \Phi_z(f)$ the graph of Φ_z is closed. Using the closed graph theorem for Frechet spaces it follows Φ_z is continuous, which exactly what we wanted. Let us apply this to the case of P being an m-homogeneous polynomial in X and $z \in monH_b(X)$. Since $P \in H_b(X)$ and the seminorms in $H_b(X)$ are given by $(\|\cdot\|_{n\cdot B_X})_{n\in\mathbb{N}}$, for some fixed $N \in \mathbb{N}$ and $\tilde{C} > 1$ we have that

$$\sum_{\alpha\in\Lambda(m,n)} |c_\alpha(P)z^\alpha| \leqslant \tilde{C}\|P\|_{N\cdot B_X} \leqslant \tilde{C}N^m\|P\|_{B_X},$$

taking $C = \tilde{C}N$ we have what we wanted to prove. □

We now have everything at hand to proceed with the proof of the upper inclusion.

Proof of the upper inclusion in Theorem 6.2.1. Fix $1 < r \leqslant 2$ and choose $z \in monH_b(\ell_r)$. Now fix n, m, choose signs as in (2.20) and define the polynomial $P(w) := \sum_{\alpha\in\Lambda(m,n)} \varepsilon_\alpha \frac{m!}{\alpha!} w^\alpha$. By Corollary 3.3.6 we know that $z^* \in monH_b(\ell_r)$. Using first the multinomial formula, then Lemma 6.2.2 and finally (2.20) we have

$$\begin{aligned}\left(\sum_{j=1}^n |z_j^*|\right)^m &= \sum_{\alpha\in\Lambda(m,n)} \frac{m!}{\alpha!}|(z^*)^\alpha| = \sum_{\alpha\in\Lambda(m,n)} \left|\varepsilon_\alpha \frac{m!}{\alpha!}(z^*)^\alpha\right| \\ &\leqslant C_{z^*}^m \sup_{u\in B_{\ell_r^n}} \left|\sum_{\alpha\in\Lambda(m,n)} \varepsilon_\alpha \frac{m!}{\alpha!} u^\alpha\right|_{\mathcal{P}(^m\ell_r^n)} \leqslant C_{z^*,r}^m (\log(m)m!n)^{1-\frac{1}{r}}.\end{aligned} \tag{6.2}$$

Taking the power $1/m$ and using Stirling's formula ($m! \leqslant \sqrt{2\pi m}e^{\frac{1}{12m}}m^m e^{-m}$) yield

$$\sum_{j=1}^n |z_j^*| \leqslant C_{z^*,r}\left[\log(m)^{\frac{1}{m}}(2\pi m)^{\frac{1}{2m}}e^{\frac{1}{12m^2}}\frac{m}{e}n^{\frac{1}{m}}\right]^{1-\frac{1}{r}}. \tag{6.3}$$

Finally, choosing $m = \lfloor \log(n+1) \rfloor$ gives that the term $\frac{1}{\log(n+1)^{1-\frac{1}{r}}} \sum_{k=1}^{n} |z_n^*|$ (for every $n \geqslant 2$) is bounded independently of n, so $z \in m_{\Psi_r}$. □

6.2.2 The lower inclusion $m_{\Psi_r} \subset monH_b(\ell_r)$

We face now the proof of the lower inclusion in Theorem 6.2.1. The main tool is the following *hypercontractive inequality* , the proof of which requires some work, that we perform all along this section.

Theorem 6.2.3. *Fix $1 < r \leqslant 2$. For every $\varepsilon > 0$ there is $C_r = C_r(\varepsilon) > 0$ such that for every $m, n \in \mathbb{N}$, every m-homogeneous polynomial in n complex variables P and every $z \in \mathbb{C}^n$, we have*

$$\sum_{\mathbf{j} \in \mathcal{J}(m,n)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}^*| \leqslant C_r(\varepsilon) m^{2+\frac{1}{r}} ((1+\varepsilon)2e)^{\frac{m}{r}} \|id : m_{\Psi_r} \to \ell_r\|^m \|z\|_{m_{\Psi_r}}^m \|P\|_{\mathcal{P}(^m\ell_r^n)} .$$

Before we start with the proof of this result, let us see how, having it at hand, we can prove the lower inclusion we are aiming at.

Proof of the lower inclusion in Theorem 6.2.1. Choose $z \in m_{\Psi_r}$ and let us see that $z \in monH_b(\ell_r)$. By Corollary 3.3.6 we may assume without loss of generality $z = z^*$. Given $f \in H_b(\ell_r)$ let us denote $P_m(f)$ for the m-homogeneous part of its Taylor expansion ($P_m(f) = \frac{d^m f}{m!}(0)$ as in Theorem 1.3.1) and Theorem 6.2.3 (with $\varepsilon = 1$) gives

$$\begin{aligned}
\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f) z^\alpha| &= \sup_{n \in \mathbb{N}} \sum_{m=0}^{\infty} \sum_{\mathbf{j} \in \mathcal{J}(m,n)} |c_{\mathbf{j}}(f) z_{\mathbf{j}}| \\
&\leqslant \sup_{n \in \mathbb{N}} \sum_{m=0}^{\infty} C_r m^{2+\frac{1}{r}} (4e)^{\frac{m}{r}} \|id\|^m \|z\|_{m_{\Psi_r}}^m \sup_{u \in B_{\ell_r^n}} \left| \sum_{\mathbf{j} \in \mathcal{J}(m,n)} c_{\mathbf{j}}(f) u_{\mathbf{j}} \right| \\
&= C_r \sum_{m=0}^{\infty} (m^{(2+\frac{1}{r})\frac{1}{m}} (4e)^{\frac{1}{r}} \|id\| \|z\|_{m_{\Psi_r}})^m \|P_m(f)\|_{\mathcal{P}(^m\ell_r)}.
\end{aligned}$$

Let us see that this sum is finite. Take $R > \sup_m \left(m^{(2+\frac{1}{r})\frac{1}{m}} (4e)^{\frac{1}{r}} \|id\| \|z\|_{m_{\Psi_r}} \right)$, then by the homogeneity of $P_m(f)$

$$\begin{aligned}
&\sum_{m=0}^{\infty} (m^{(2+\frac{1}{r})\frac{1}{m}} (4e)^{\frac{1}{r}} \|id\| \|z\|_{m_{\Psi_r}})^m \|P_m(f)\|_{\mathcal{P}(^m\ell_r)} \\
&\quad = \sum_{m=0}^{\infty} \left(\frac{m^{(2+\frac{1}{r})\frac{1}{m}} (4e)^{\frac{1}{r}} \|id\| \|z\|_{m_{\Psi_r}}}{R} \right)^m \sup_{w \in R \cdot B_{\ell_r}} |P_m(f)(w)| \\
&\quad \leqslant \sum_{m=0}^{\infty} \left(\frac{m^{(2+\frac{1}{r})\frac{1}{m}} (4e)^{\frac{1}{r}} \|id\| \|z\|_{m_{\Psi_r}}}{R} \right)^m \sup_{w \in R \cdot B_{\ell_r}} |f(w)| < \infty,
\end{aligned}$$

where the last step is due to Cauchy's inequality. This completes the proof. □

We start now the way to the proof of Theorem 6.2.3. We begin with a simple remark.

Remark 6.2.4. If $z \in m_{\Psi_r}$, then

$$n|z_n^*| \leqslant \sum_{l=1}^{n} z_l^* \leqslant \|z\|_{m_{\Psi_r}} \log(n+1)^{\frac{1}{r'}}.$$

That is

$$|z_n^*| \leqslant \|z\|_{m_{\Psi_r}} \frac{\log(n+1)^{\frac{1}{r'}}}{n}$$

for every $n \in \mathbb{N}$. This gives

$$\sum_{j=1}^{n} |z_j|^r \leqslant \sum_{j=1}^{n} |z_j^*|^r \leqslant \|z\|_{m_{\Psi_r}}^r \sum_{j=1}^{n} \frac{\log(j+1)^{\frac{r}{r'}}}{j^r}.$$

This implies $\|id : m_{\Psi_r} \to \ell_r\| \leqslant \left(\sum_{j=1}^{\infty} \frac{\log(j+1)^{\frac{r}{r'}}}{j^r}\right)^{1/r}$ (note that this series is convergent for $1 < r$).

The *Bayart-Defant-Schlüters inequality* in Theorem 2.1.7 will be fundamental here as it has been proving Theorem 3.4.1 and Theorem 5.2.1 in the case $1 < q < p \leqslant 2$.

Lemma 6.2.5. *Let $1 < r \leqslant 2$, there is $A_r > 0$ such that for every $m, n \in \mathbb{N}$, every $P \in \mathcal{P}(^m\mathbb{C}^n)$ and every decreasing $z \in \mathbb{C}^n$ we have*

$$\sum_{\mathbf{j} \in \mathcal{J}(m,n)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}| \leqslant A_r m^{1+\frac{1}{r}} e^{\frac{m}{r}} \|z\|_{m_{\Psi_r}}^2 \left(\sum_{k=1}^{n} \frac{\log(k+1)^{\frac{2}{r'}}}{k^{1+\frac{1}{r'}}} \sum_{\mathbf{i} \in \mathcal{J}(m-2,k)} |z_{\mathbf{i}}| |\mathbf{i}|^{\frac{1}{r}} \right) \|P\|_{\mathcal{P}(^m\ell_r^n)}.$$

Proof. Consider $P = \sum_{\mathbf{j} \in \mathcal{J}(m,n)} c_{\mathbf{j}}(P) z_{\mathbf{j}} \in \mathcal{P}(^m\mathbb{C}^n)$ and $z \in \mathbb{C}^n$ decreasing. Using first Hölder's inequality and then (2.8) we have

$$\begin{aligned}
\sum_{\mathbf{j} \in \mathcal{J}(m,n)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}| &= \sum_{\mathbf{j} \in \mathcal{J}(m-1,n)} \sum_{j_m = j_{m-1}}^{n} |c_{(\mathbf{j},j_m)}(P) z_{\mathbf{j}} z_{j_m}| \\
&\leqslant \sum_{\mathbf{j} \in \mathcal{J}(m-1,n)} |z_{\mathbf{j}}| \Big(\sum_{j_m = j_{m-1}}^{n} |c_{(\mathbf{j},j_m)}(P)|^{r'} \Big)^{\frac{1}{r'}} \Big(\sum_{j_m = j_{m-1}}^{n} |z_{j_m}|^r \Big)^{\frac{1}{r}} \\
&\leqslant e^{1-\frac{1}{r}} m e^{\frac{m}{r}} \|P\|_{\mathcal{P}(^m\ell_r)} \sum_{\mathbf{j} \in \mathcal{J}(m-1,n)} |z_{\mathbf{j}}| |\mathbf{j}|^{\frac{1}{r}} \Big(\sum_{j_m = j_{m-1}}^{n} |z_{j_m}|^r \Big)^{\frac{1}{r}} \\
&= e^{1-\frac{1}{r}} m e^{\frac{m}{r}} \|P\|_{\mathcal{P}(^m\ell_r)} \sum_{j_{m-1}=1}^{n} |z_{j_{m-1}}| \sum_{\mathbf{i} \in \mathcal{J}(m-2,j_{m-1})} |z_{\mathbf{i}}| |(\mathbf{i}, j_{m-1})|^{\frac{1}{r}} \Big(\sum_{j_m = j_{m-1}}^{n} |z_{j_m}|^r \Big)^{\frac{1}{r}} \\
&\leqslant e^{1-\frac{1}{r}} m e^{\frac{m}{r}} \|P\|_{\mathcal{P}(^m\ell_r)} (m-1)^{\frac{1}{r}} \sum_{j_{m-1}=1}^{n} |z_{j_{m-1}}| \Big(\sum_{j_m = j_{m-1}}^{n} |z_{j_m}|^r \Big)^{\frac{1}{r}} \sum_{\mathbf{i} \in \mathcal{J}(m-2,j_{m-1})} |z_{\mathbf{i}}| |\mathbf{i}|^{\frac{1}{r}},
\end{aligned} \tag{6.4}$$

where the last inequality is due to the fact that $|(\mathbf{i}, j_{m-1})| \leqslant (m-1)|\mathbf{i}|$ for every $\mathbf{i} \in \mathcal{J}(m-2, j_{m-1})$.

We now bound the factor $|z_{j_{m-1}}|\Big(\sum_{j_m=j_{m-1}}^{n} |z_{j_m}|^r\Big)^{\frac{1}{r}}$. For each $1 \leqslant j \leqslant n$ we use Remark 6.2.4 to obtain (note that $\frac{r}{r'} - 1 = r - 2 \leqslant 0$).

$$|z_j|\Big(\sum_{k=j}^{n} |z_k|^r\Big)^{\frac{1}{r}} \leqslant \|z\|_{m_{\Psi_r}}^2 \frac{\log(j+1)^{\frac{1}{r'}}}{j}\Big(\sum_{k=j}^{n} \frac{\log(k+1)^{\frac{r}{r'}}}{k^r}\Big)^{\frac{1}{r}}$$
$$\leqslant \|z\|_{m_{\Psi_r}}^2 \frac{\log(j+1)^{\frac{1}{r'}}}{j} \log(j+1)^{\frac{1}{r'}-\frac{1}{r}}\Big(\sum_{k=j}^{n} \frac{\log(k+1)}{k^r}\Big)^{\frac{1}{r}}.$$

We deal with the last sum

$$\sum_{k=j}^{n} \frac{\log(k+1)}{k^r} \leqslant \Big(1+\frac{1}{j}\Big)^r \sum_{k=j}^{n} \frac{\log(k+1)}{(k+1)^r} \leqslant 2^r \sum_{k=j+1}^{n+1} \frac{\log(k)}{k^r} \leqslant 2^{r+2}\int_j^{n+1} \frac{\log(x)}{x^r}dx$$
$$\leqslant 2^{r+2}\frac{(r-1)\log(j)+1}{(r-1)^2 j^{r-1}} \leqslant 2^{r+2}\frac{2r}{(r-1)^2}\frac{\log(j+1)}{j^{r-1}},$$

and

$$|z_j|\Big(\sum_{k=j}^{n} |z_k|^r\Big)^{\frac{1}{r}} \leqslant 2^{r+2}\frac{2r}{(r-1)^2}\|z\|_{m_{\Psi_r}}^2 \frac{\log(j+1)^{\frac{2}{r'}}}{j^{1+\frac{1}{r'}}}$$

This inequality together with (6.4) give us the desired conclusion. □

In view of Lemma 6.2.5, now we need to bound $\sum_{\mathbf{i}\in\mathcal{J}(m-2,k)} |z_{\mathbf{i}}||\mathbf{i}|^{\frac{1}{r}}$ in a suitable way (depending on k). To this purpose we switch to the α-notation of multi-indices for convenience, then the sum reads

$$\sum_{\alpha\in\Lambda(m-2,k)} |z|^{\alpha}|[\alpha]|^{1/r}. \tag{6.5}$$

The strategy is to analyze smaller pieces of the sum: the *tetrahedral* and an *even* part introduced in Section 6.1, and use the bounds obtained for each of these parts to conclude something about sums which involve general monomials.

Lemma 6.2.6. *Fixed* $1 < r \leqslant 2$ *and* $M, N \in \mathbb{N}$, *for every decreasing* $z \in \mathbb{C}^N$ *we have*

$$\sum_{\alpha\in\Lambda_T(M,N)} |z^{\alpha}||[\alpha]|^{\frac{1}{r}} \leqslant 2(1+\varepsilon)^{\frac{M}{r'}}\|z\|_{m_{\Psi_r}}^M N^{\frac{1}{(1+\varepsilon)r'}},$$

for every $\varepsilon > 0$ *and*

$$\sum_{\alpha\in\Lambda_E(M,N)} |z^{\alpha}||[\alpha]|^{\frac{1}{r}} \leqslant \|z\|_{\ell_r}^M \leqslant \|id : m_{\Psi_r} \to \ell_r\|^M \|z\|_{m_{\Psi_r}}^M.$$

Proof. We begin with the first inequality, observing that it is obvious if $N = 1$. We may, then, assume $N \geqslant 2$. Then, given $\alpha \in \Lambda_T(M,N)$, note that $\alpha! = 1$ and $|[\alpha]|$ is exactly $M!$. Then,

$$\sum_{\alpha\in\Lambda_T(M,N)} |z^\alpha||[\alpha]|^{\frac{1}{r}} = \sum_{\alpha\in\Lambda_T(M,N)} |z^\alpha||[\alpha]|\frac{1}{|[\alpha]|^{\frac{1}{r'}}} \leqslant \Big(\sum_{k=1}^N |z_k|\Big)^M \frac{1}{M!^{\frac{1}{r'}}}$$
$$\leqslant \|z\|_{m_{\Psi_r}}^M \log(N+1)^{\frac{M}{r'}} \frac{1}{M!^{\frac{1}{r'}}} \leqslant 2\|z\|_{m_{\Psi_r}}^M \Big(\frac{\log(N)^M}{M!}\Big)^{\frac{1}{r'}}.$$

A simple calculus argument shows that the function $f : [1,\infty[\to \mathbb{R}$ given by $f(x) = \frac{\log(x)^M}{x^{1/(1+\varepsilon)}}$ is bounded by $\big(\frac{(1+\varepsilon)M}{e}\big)^M$, then $\log(N)^M \leqslant N^{1/(1+\varepsilon)}\big(\frac{(1+\varepsilon)M}{e}\big)^M$. On the other hand $M! \geqslant \big(\frac{M}{e}\big)^M$.This gives the conclusion.
For the proof of the second inequality let us recall first that for each $\alpha \in \Lambda_E(M,N)$ there is a unique $\beta \in \Lambda(M/2,N)$ such that $\alpha = 2\beta$ and, moreover,

$$|[\alpha]| = \frac{M!}{\alpha_1!\cdots\alpha_N!} = \Big(\frac{(M/2)!}{\beta_1!\cdots\beta_N!}\Big)^2 \frac{M!}{(M/2)!(M/2)!}\prod_{i=1}^N \frac{\beta_i!\beta_i!}{(2\beta_i)!} \leqslant |[\beta]|^2,$$

where last inequality holds because $2^k \leqslant \frac{(2k)!}{k!^2} \leqslant 2^{2k}$ and then

$$\frac{M!}{(M/2)!(M/2)!}\prod_{i=1}^N \frac{\beta_i!\beta_i!}{(2\beta_i)!} \leqslant 2^M \prod_{i=1}^N \frac{1}{2^{\beta_i}} = 1.$$

Then (note that, since $2/r \geqslant 1$, the ℓ_1 norm bounds the $\ell_{2/r}$ norm)

$$\sum_{\alpha\in\Lambda_E(M,N)} |z^\alpha||[\alpha]|^{\frac{1}{r}} \leqslant \sum_{\beta\in\Lambda(M/2,N)} |(z^2)^\beta||[\beta]|^{2/r} = \sum_{\beta\in\Lambda(M/2,N)} \Big(|(z^r)^\beta||[\beta]|\Big)^{2/r}$$
$$\leqslant \Big(\sum_{\beta\in\Lambda(M/2,N)} |(z^r)^\beta||[\beta]|\Big)^{2/r} = \Big(\sum_{l=1}^N |z_l|^r\Big)^{M/r} \leqslant \|id : m_{\Psi_r} \to \ell_r\|^M \|z\|_{m_{\Psi_r}}^M.$$

□

Lemma 6.2.7. *Given $1 < r \leqslant 2$ there is a constant $K_r \geqslant 1$ such that for every $M, N \in \mathbb{N}$, and every decreasing $z \in \mathbb{C}^N$ we have*

$$\sum_{\alpha\in\Lambda(M,N)} |z^\alpha||[\alpha]|^{\frac{1}{r}} \leqslant K_r(M+1)(1+\varepsilon)^{\frac{M}{r'}} 2^{\frac{M}{r}+1} N^{\frac{1}{(1+\varepsilon)r'}} (\|id : m_{\Psi_r} \to \ell_r\| \|z\|_{m_{\Psi_r}})^M,$$

for every $\varepsilon > 0$.

Proof. Choose some decreasing z and use Remark 6.1.1 and Lemma 6.2.6 to get

$$\begin{aligned}
\sum_{\alpha\in\Lambda(M,N)} |z^\alpha||[\alpha]|^{\frac{1}{r}} &= \sum_{k=0}^{M} \sum_{\alpha_T\in\Lambda_T(k,N)} \sum_{\alpha_E\in\Lambda_E(M-k,N)} |z^{(\alpha_T+\alpha_E)}||[\alpha_T+\alpha_E]|^{\frac{1}{r}} \\
&\leqslant 2^{\frac{M}{r}} \sum_{k=0}^{M} \left(\sum_{\alpha_T\in\Lambda_T(k,N)} |z_T^{\alpha}||[\alpha_T]|^{\frac{1}{r}} \right) \left(\sum_{\alpha_E\in\Lambda_E(M-k,N)} |z_E^{\alpha}||[\alpha_E]|^{\frac{1}{r}} \right) \\
&\leqslant 2^{\frac{M}{r}} \sum_{k=0}^{M} \left((1+\varepsilon)^{\frac{k}{r'}} \|z\|_{m_{\Psi_r}}^k N^{\frac{1}{(1+\varepsilon)r'}} \right) \left(\|id : m_{\Psi_r} \to \ell_r\|^{M-k} \|z\|_{m_{\Psi_r}}^{M-k} \right) \\
&\leqslant 2^{\frac{M}{r}+1} (1+\varepsilon)^M \|id : m_{\Psi_r} \to \ell_r\|^M \|z\|_{m_{\Psi_r}}^M N^{\frac{1}{(1+\varepsilon)r'}} \sum_{k=0}^{M} 2^{k(1-\frac{2}{r})}.
\end{aligned}$$

For $r = 2$ the last sum is exactly $M+1$. If $1 < r < 2$ the sum is bounded by $\frac{2^{2/r}}{2^{2/r}-2}$. This completes the proof. ☐

We are finally in the position to give the proof of Theorem 6.2.3 from which (as we already saw) the lower inclusion in Theorem 6.2.1 follows.

Proof of Theorem 6.2.3. Fix $1 < r \leqslant 2$ and n, m. Pick then $P \in P \in \mathcal{P}(^m\mathbb{C}^n)$ and $z \in \mathbb{C}^n$. Since $\|z\|_{m_{\Psi_r}} = \|z^*\|_{m_{\Psi_r}}$, we may assume $z = z^*$. Applying Lemma 6.2.7 with $M = m-2$ and $N = k$ after Lemma 6.2.5 yields

$$\begin{aligned}
&\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}| \leqslant 2A_r m^{1+\frac{1}{r}} e^{\frac{m}{r}} \|z\|_{m_{\Psi_r}}^2 \\
&\quad \times \left(\sum_{k=1}^{n} \frac{\log(k+1)^{\frac{2}{r'}}}{k^{1+\frac{1}{r'}}} K_r (m-1) k^{\frac{1}{(1+\varepsilon)r'}} \left((2(1+\varepsilon))^{1/r'} \|id\| \|z\|_{m_{\Psi_r}}\right)^{m-2} \right) \|P\|_{\mathcal{P}(^m\ell_r^n)} \\
&\qquad \leqslant 2A_r K_r m^{2+\frac{1}{r}} ((1+\varepsilon)2e)^{\frac{m}{r}} \|id\|^m \|z\|_{m_{\Psi_r}}^m \left(\sum_{k=1}^{n} \frac{\log(k+1)^{\frac{2}{r'}}}{k^{1+\frac{\varepsilon}{(1+\varepsilon)r'}}} \right) \|P\|_{\mathcal{P}(^m\ell_r^n)}.
\end{aligned}$$

Since $r > 1$ the series $\sum_{k=1}^{\infty} \frac{\log(k+1)^{\frac{2}{r'}}}{k^{1+\frac{\varepsilon}{(1+\varepsilon)r'}}}$ is convergent. This completes the proof. ☐

6.3 The case $2 < r \leqslant \infty$.

Now we concentrate in the case $2 < r \leqslant \infty$. We will study the more general case of $monH_b(\ell_{r,s})$ with $2 < r, s \leqslant \infty$. We will be able to characterize that set of monomial convergence in the case $s = \infty$ for every $2 < r \leqslant \infty$ and to give very tight lower and upper bound for that sets for the remaining cases.

To do so let us define a useful family of Banach sequence spaces that generalize the *Marcinkiewicz spaces*. Let $\Psi = (\Psi(n))_{n=0}^{\infty}$ be a symbol, i.e., an increasing sequence of nonnegative real numbers with $\Psi(0) = 0$ and $\Psi(n) > 0$ for every $n \in \mathbb{N}$. For $1 \leqslant r, s \leqslant \infty$ we define $X_{r,s}(\Psi)$ as

$$X_{r,s}(\Psi) := \left\{ z \in \ell_\infty : \sup_{n \geqslant 1} \frac{1}{\Psi(n)} \|(z_k^*)_{k=1}^n\|_{\ell_{r,s}^n} < \infty \right\},$$

endowed with the norm

$$\|z\|_{X_{r,s}(\Psi)} := \sup_{n \geqslant 1} \frac{\|(z_k^*)_{k=1}^n\|_{\ell_{r,s}^n}}{\Psi(n)}.$$

Whenever $r = s$ we will simply write $X_r(\Psi)$ for $X_{r,s}(\Psi)$.

Remark 6.3.1. For every symbol Ψ, real numbers $1 \leqslant r, s \leqslant \infty$ and $z \in X_{r,s}(\Psi)$ and using $n^{\frac{1}{r}} \sim \left(\sum_{k=1}^n k^{\frac{s}{r}-1}\right)^{1/s}$ we have the following chain of inequalities

$$|z_n^*| n^{\frac{1}{r}} \leqslant C_1 |z_n^*| \left(\sum_{k=1}^n k^{\frac{s}{r}-1} \right)^{1/s} \leqslant C_1 \|(z_k^*)_{k=1}^n\|_{\ell_{r,s}^n} \ll \|z\|_{X_{r,s}(\Psi)} \Psi(n),$$

then $|z_n^*| \ll \|z\|_{X_{r,s}(\Psi)} \frac{\Psi(n)}{n^{1/r}}$.

It will be useful to consider the mapping $\varphi : [2, \infty] \to [1, 2]$ such that $\varphi(r) = \left(\frac{1}{2} + \frac{1}{r}\right)^{-1}$ whenever $r \in [2, \infty)$ and $\varphi(\infty) = 2$. Observe that for some fixed $2 \leqslant r \leqslant \infty$ the conjugate exponent for $\varphi(r)$ is $\varphi(r)' = \left(\frac{1}{2} - \frac{1}{r}\right)^{-1}$ $(1 = \frac{1}{\varphi(r)} + \frac{1}{\varphi(r)'})$.

Theorem 6.3.2. *Given $2 < r \leqslant \infty$ and $2 < s \leqslant \infty$, for any $\frac{1}{s} < \delta < \frac{1}{2}$ it holds*

$$X_{\varphi(r),\varphi(s)}(\Phi^\delta) \subset monH_b(\ell_{r,s}) \subset X_{\varphi(r),\varphi(s)}(\Psi_2),$$

where $\Psi_2 = (\sqrt{\log(n+1)})_{n=0}^\infty$ (as defined in (6.1)) and Φ^δ defined on $n \in \mathbb{N}_0$ as

$$\Phi^\delta(n) = \begin{cases} \Psi_2(n) & \text{for } s = \infty \\ \log(n+1)^{\frac{1}{2}-\delta} & \text{for } s < \infty. \end{cases}$$

The upper bound holds for every $2 \leqslant r, s \leqslant \infty$ (including $s = 2$).

The proof of Theorem 6.3.2 requires some work, in particular we will divide it into the upper and lower bound. We prove the upper bound along Section 6.3.1 and the lower bound in Section 6.3.2. First let us discuss some of the corollaries and consequences of this theorem.

Notice that for $2 < s < \infty$ and $1/s < \delta < 1/2$ the symbol Φ^δ is close to Ψ_2. More precisely, given $n \in \mathbb{N}$ it holds

$$\Phi^\delta(n) = \Psi_2(n) \ \log(n+1)^{-\delta}.$$

Also for $s = \infty$ we have $\Phi^\delta = \Psi_2$ not depending on δ, this shows that, assuming Theorem 6.3.2, the following corollary holds.

Corollary 6.3.3. *Given $2 \leqslant r \leqslant \infty$ it holds*

$$monH_b(\ell_{r,\infty}) = X_{\varphi(r),2}(\Psi_2) = \left\{ z \in \ell_\infty : \sup_{n \geqslant 1} \frac{1}{\sqrt{\log(n+1)}} \left(\sum_{k=1}^{n} k^{\frac{2}{r}} |z_k^*|^2 \right)^{1/2} < \infty \right\}.$$

In particular Corollary 6.3.3 implies

$$monH_b(\ell_\infty) = X_2(\Psi_2) = \left\{ z \in \ell_\infty : \sup_{n \geqslant 1} \frac{1}{\sqrt{\log(n+1)}} \left(\sum_{k=1}^{n} |z_k^*|^2 \right)^{1/2} < \infty \right\}.$$

This result looks very similar to Theorem 3.2.3. Even more, we may consider the norm in $z \in \mathbb{C}^{\mathbb{N}}$ given by

$$\|z\|_Z := \max \left\{ \limsup_{n \to \infty} \frac{1}{\sqrt{\log(n)}} \left(\sum_{k=1}^{n} (z_k^*)^2 \right)^{1/2}, \|z\|_{\ell_\infty} \right\}.$$

It is easy too see

$$monH_b(\ell_\infty) = X_2(\Psi_2) = \{z \in \mathbb{C}^{\mathbb{N}} : \|z\|_Z < \infty\},$$

as sets, and by Theorem 3.2.3 it holds

$$B_Z \subset monH_\infty(B_{\ell_\infty}) \subset \overline{B_Z}. \tag{6.6}$$

In a way, with the right norm, we can write then

$$B_{monH_b(\ell_\infty)} \subset monH_\infty(B_{\ell_\infty}) \subset \overline{B_{monH_b(\ell_\infty)}}.$$

Remark 6.3.4. For a fixed symbol Ψ we have $X_1(\Psi) = m_\Psi$. In other word, these new families of Banach sequence spaces generalize the Marcinkiewicz spaces. In particular it holds

$$m_{\Psi_2} = X_1(\Psi_2). \tag{6.7}$$

Now we are ready to begin the proof of the upper inclusion in Theorem 6.3.2.

6.3.1 The upper inclusion $monH_b(\ell_{r,s}) \subset X_{\varphi(r),\varphi(s)}(\Psi_2)$.

Here we give a proof for the upper inclusion on Theorem 6.3.2. Our arguments differ from the classic techniques. For example, in the previous section we have used Bayart's polynomials from (2.20). In this case we choose a strategy that is more reminiscent to the one used in [DF11, BDS19]. In those articles the authors found sharp upper bounds for the unconditionality constant of $\mathcal{P}(^m\ell_p^n)$ with $2 \leqslant p < \infty$ by comparing them with the one of $\mathcal{P}(^m\ell_\infty^n)$, which they managed to calculate. Here we imitate that heuristic to develop a tool that allow us to link $monH_b(\ell_{p,s})$ with $monH_b(\ell_2)$ which we already know thanks to Theorem 6.2.1.

Now we need the following well known *Hardy-Littlewood rearrangement inequality* (see for example [HLP52, Section 10.2, Theorem 368]).

Lemma 6.3.5 (Hardy-Littlewood rearrangement inequality)**.** *Let* $(a_k)_{k\in\mathbb{N}}$ *and* $(b_k)_{k\in\mathbb{N}}$ *two non-increasing sequences of non-negative real numbers. Then, for every* $m \in \mathbb{N}$ *and every injection* $\sigma : \mathbb{N} \to \mathbb{N}$ *we have*

$$\sum_{k=1}^{m} a_{\sigma(k)} b_k \leqslant \sum_{k=1}^{m} a_k b_k.$$

The next technical lemma is the first step in the direction of proving the upper inclusion.

Lemma 6.3.6. *Given* $2 \leqslant r, s \leqslant \infty$ *and* $\xi \in \ell_{\varphi(r)',\varphi(s)'}$ *the linear operator*

$$\begin{aligned} \ell_{r,s} &\xrightarrow{D_\xi} \ell_2 \\ (z_j)_{j\geqslant 1} &\longmapsto (z_j \xi_j)_{j\geqslant 1}, \end{aligned}$$

is well defined and bounded. Moreover, $\|D_\xi\| \leqslant \|\xi\|_{\ell_{\varphi(r)',\varphi(s)'}}$.

Proof. Let us take $\xi \in \ell_{\varphi(r)',\varphi(s)'}$ and $z \in \ell_{r,s}$, then

$$\begin{aligned} \|D_\xi(z)\|_{\ell_2} &= \left(\sum_{k\geqslant 1} |\xi_k z_k|^2 \right)^{1/2} \\ \text{(by Lemma 6.3.5)} &\leqslant \left(\sum_{k\geqslant 1} |\xi_k^* z_k^*|^2 \right)^{1/2} \\ &= \left(\sum_{k\geqslant 1} |\xi_k^* z_k^*|^2 k^t k^{-t} \right)^{1/2}, \end{aligned}$$

where $t = 2(1/r - 1/s)$. Now, by Hölder inequality (since $s \geqslant 2$), it holds

$$\left(\sum_{k\geqslant 1} |\xi_k^* z_k^*|^2 k^t k^{-t} \right)^{1/2} \leqslant \left(\sum_{k\geqslant 1} |z_k^*|^s k^{\frac{ts}{2}} \right)^{1/s} \left(\sum_{k\geqslant 1} |\xi_k^*|^{\frac{2s}{s-2}} k^{\frac{-ts}{s-2}} \right)^{\frac{s-2}{2s}},$$

as $\frac{ts}{2} = \frac{s}{r} - 1$ and $\frac{-ts}{s-2} = \frac{\varphi(s)'}{\varphi(r)'} - 1$ the last term in the previous chain of inequalities equals to $\|z\|_{\ell_{r,s}} \|\xi\|_{\ell_{\varphi(r)',\varphi(s)'}}$, so

$$\|D_\xi(z)\|_{\ell_2} \leqslant \|z\|_{\ell_{r,s}} \|\xi\|_{\ell_{\varphi(r)',\varphi(s)'}}. \tag{6.8}$$

When $2 \leqslant r < s \leqslant \infty$ we need to consider the norm in $\ell_{r,s}$ given by $\|\cdot\|_{\ell_{(r,s)}}$, using (1.2) we have $\|z\|_{\ell_{r,s}} \leqslant \|z\|_{\ell_{(r,s)}}$. Thus by the bound in (6.8), for every $2 \leqslant r, s \leqslant \infty$ we have $\|D_\xi\| \leqslant \|\xi\|_{\ell_{\varphi(r)',\varphi(s)'}}$. □

Now we are able to show the upper inclusion in Theorem 6.3.2.

Proof of the upper inclusion in Theorem 6.3.2. For the case $r = s = 2$, Theorem 6.2.1 and Remark 6.3.4 with symbol Ψ_2 do the job.

In other cases, take $z \in monH_b(\ell_{r,s})$, we will see that for every $\xi \in \ell_{\varphi(r)',\varphi(s)'}$ it holds $D_\xi z \in monH_b(\ell_2) \subset X_1(\Psi_2)$. Given $f \in H_b(\ell_2)$ we define $g_\xi = f \circ D_\xi \in H_b(\ell_{r,s})$, by Remark 3.1.5 it holds $c_\alpha(g_\xi) = \xi^\alpha c_\alpha(f)$. Now for every $f \in H_b(\ell_2)$ we have

$$\begin{aligned}\sum_{m\geqslant 0}\sum_{\alpha\in\Lambda(m,n)} |c_\alpha(f)(D_\xi z)^\alpha| &= \sum_{m\geqslant 0}\sum_{\alpha\in\Lambda(m,n)} |c_\alpha(f)\xi^\alpha z^\alpha| \\ &= \sum_{m\geqslant 0}\sum_{\alpha\in\Lambda(m,n)} |c_\alpha(g)z^\alpha| < \infty,\end{aligned}$$

so $D_\xi z \in monH_b(\ell_2)$.

This induces the operator

$$\begin{aligned} T_z : \ell_{\varphi(r)',\varphi(s)'} &\to X_1(\Psi_2) &(6.9)\\ \xi &\mapsto \xi z, &(6.10)\end{aligned}$$

which turns out to be bounded by the Closed graph theorem. Let us show the last assertion is true. Let $(\xi^N)_{N\geqslant 1} \subset \ell_{\varphi(p)',\varphi(q)'}$ be a sequence such that

$$\begin{aligned} \|\xi^N - \xi\|_{\ell_{\varphi(r)',\varphi(s)'}} &\to 0 &(6.11)\\ \|T_z(\xi^N) - w\|_{X_1(\Psi_2)} &\to 0, &(6.12)\end{aligned}$$

as $N \to \infty$. We need to show $w = T_z(\xi) = \xi \cdot z$. By equation (6.11) we have

$$\sum_{k\geqslant 1} k^{(\frac{1}{s}-\frac{1}{r})\varphi(s)'} |(\xi^N - \xi)^*_k|^{\varphi(s)'} \leqslant A_N,$$

where $A_N \to 0$, then for every fixed $k \in \mathbb{N}$ it follows

$$k^{(\frac{1}{s}-\frac{1}{r})\varphi(s)'} |(\xi^N - \xi)^*_k|^{\varphi(s)'} \leqslant A_N.$$

Since $|\xi^N_k - \xi_k| \leqslant |(\xi^N - \xi)^*_k| \to 0$ then for any fixed k $\xi^N_k \to \xi_k$ as $N \to \infty$. Analogously, equation (6.12) implies $\sup_{n\geqslant 1} \frac{1}{\sqrt{\log(n+1)}} \sum_{k=1}^n |(\xi^N \cdot z - w)^*_k| \leqslant B_N$ where $B_N \to 0$. Now for some fixed $k \in \mathbb{N}$ and taking $n \geqslant k$ we have

$$\sum_{k=1}^n |(\xi^N \cdot z - w)^*_k| \leqslant B_N \sqrt{\log(n+1)},$$

then, as before $\xi^N_k z_k \to w_k$. Finally $w_k = \xi_k z_k$, so $w = T_z(\xi)$ as we wanted.

Now, being T_z bounded, it follows

$$\begin{aligned}
\|T_z\| &= \sup_{\xi \in B_{\ell_{\varphi(r)',\varphi(s)'}}} \|\xi z\|_{X_1(\Psi_2)} \\
&= \sup_{\xi \in B_{\ell_{\varphi(r)',\varphi(s)'}}} \sup_{n \geqslant 1} \frac{1}{\sqrt{\log(n+1)}} \sum_{k=1}^{n} |(\xi_k z_k)^*| \\
&= \sup_{n \geqslant 1} \frac{1}{\sqrt{\log(n+1)}} \sup_{\xi \in B_{\ell_{\varphi(r)',\varphi(s)'}}} \left| \sum_{k=1}^{n} \xi_k z_k^* \right| \\
&\sim \sup_{n \geqslant 1} \frac{1}{\sqrt{\log(n+1)}} \|(z_k^*)_{k=1}^n\|_{\ell_{\varphi(r),\varphi(s)}} \\
&= \|z\|_{X_{\varphi(r),\varphi(s)}(\Psi_2)},
\end{aligned}$$

since, using $1 < \varphi(r)$ by Theorem 1.1.4 the space $(\ell^n_{\varphi(r),\varphi(r)})'$ and $\ell^n_{\varphi(r)',\varphi(s)'}$ are isomorphic for every $n \in \mathbb{N}$ (and the norm of the isomorphism does not depend on n). Then, as $\|T_z\| \sim \|z\|_{X_{\varphi(p),\varphi(q)}(\Psi_2)}$ is a finite number $z \in X_{\varphi(p),\varphi(q)}(\Psi_2)$. □

6.3.2 The lower inclusion $X_{\varphi(r),\varphi(s)}(\Phi^\delta) \subset monH_b(\ell_{r,s})$.

We face now the proof of the lower inclusion in Theorem 6.3.2. The main tool is the following result, whose proof is performed all along this section.

Theorem 6.3.7. *Fix $2 < r \leqslant \infty$ and $2 < s \leqslant \infty$. Let $1/s < \delta < 1/2$ and define Φ^δ as in Theorem 6.3.2. For every $\varepsilon > 0$ and $m, n \in \mathbb{N}$, every $P \in \mathcal{P}(^m\mathbb{C}^n)$ and every $z \in \mathbb{C}^n$, we have*

$$\sum_{\mathbf{j} \in \mathcal{J}(m,n)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}^*| \leqslant D(\varepsilon) e m^{3/2} (1+\varepsilon)^{(m-1)/2} A^{m-1} \|P\|_{\mathcal{P}(^m\ell^n_{r,\infty})} \|z\|^m_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)},$$

where $A = 2C^{-1}_{r,s,\delta} \left(\sum_{l=1}^{\infty} \left(\frac{\log(l+1)}{l} \right)^2 \right)^{1/4}$ and $C_{r,s,\delta}, D(\varepsilon) > 0$ are constants not depending on m nor n.

Before starting the proof of this result, let us see how, using it, we can show the lower inclusion.

Proof of the lower inclusion in Theorem 6.3.2. Choose $z \in X_{\varphi(r),\varphi(s)}(\Phi^\delta)$ and let us see that $z \in monH_b(\ell_{r,s})$. By Corollary 3.3.6 we may assume without loss of generality $z = z^*$. Given $f \in H_b(\ell_{r,s})$ (recall that we denote $P_m(f)$ for the m-homogeneous part of its Taylor expansion) and Theorem 6.3.7 (with $\varepsilon = 1$) gives

$$\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f)z^\alpha| = \sup_{n\in\mathbb{N}} \sum_{m=0}^{\infty} \sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(f)z_{\mathbf{j}}|$$

$$\leqslant \sup_{n\in\mathbb{N}} \sum_{m=0}^{\infty} D(\varepsilon)em^{3/2}(\sqrt{2}A)^{m-1}\|z\|^m_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)} \sup_{u\in B_{\ell_r^n}} \left|\sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}}(f)u_{\mathbf{j}}\right|$$

$$= D(\varepsilon)e \sum_{m=0}^{\infty} m^{3/2}(\sqrt{2}A)^{m-1}\|z\|^m_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)}\|P_m(f)\|_{\mathcal{P}(^m\ell^n_{r,\infty})}.$$

Let us see that this sum is finite. Take

$$R > S := \sup_{m\geqslant 1}\left(m^{3/2}(\sqrt{2}A)^{m-1}\|z\|^m_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)}\right)^{1/m},$$

then by the homogeneity of $P_m(f)$ it follows

$$\sum_{m=0}^{\infty} m^{3/2}(\sqrt{2}A)^{m-1}\|z\|^m_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)}\|P_m(f)\|_{\mathcal{P}(^m\ell_r)}$$

$$= \sum_{m=0}^{\infty} \frac{m^{3/2}(\sqrt{2}A)^{m-1}\|z\|^m_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)}}{R^m} \sup_{w\in R\cdot B_{\ell_r}} |P_m(f)(w)|$$

$$\leqslant \sum_{m=0}^{\infty}\left(\frac{S}{R}\right)^m \sup_{w\in R\cdot B_{\ell_r}} |f(w)| < \infty,$$

where the last step is due to Cauchy's inequality. This completes the proof. □

In the previous section we used Theorem 2.1.7 to achieve the lower bounds, here we will replace it with Theorem 2.1.9. Thanks to Remark 3.1.5 we will manage to use it to compare a mixed coefficient norm of a given polynomial with its uniform norm on $\ell_{r,s}$. If we had a new mixed inequality in the fashion of Theorem 2.1.7 and Theorem 2.1.9 that fit our problem better, perhaps we could close the gap between the upper and lower bound in the Theorem 6.3.2.

Lemma 6.3.8. *Fix $2 \leqslant r \leqslant \infty$ and $2 < s \leqslant \infty$. Let $1/s < \delta < 1/2$ and define Φ^δ as in Theorem 6.3.2. For every $m, n \in \mathbb{N}$, every $P \in \mathcal{P}(^m\mathbb{C}^n)$ and every decreasing $z \in \mathbb{C}^n$ we have*

$$\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P)z_{\mathbf{j}}| \leqslant em2^{m-1}\|P\|_{\mathcal{P}(^m\ell^n_{r,\infty})}\|z\|_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)} \sup_{k=1,\dots,n} \frac{\Phi^\delta(k)}{k^{1/\varphi(r)}|w_k|}\left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,k)}\left|\frac{z_{\mathbf{j}}}{w_{\mathbf{j}}}\right|^2\right)^{\frac{1}{2}},$$

where $w \in B_{\ell^n_{r,s}}$.

Proof. Consider $P \in \mathcal{P}(^m\mathbb{C}^n)$ and $z \in \mathbb{C}^n$ decreasing. Using first Hölder's inequality, Theorem 2.1.9 and Remark 3.1.5 with $\mathcal{R} = B_{\ell^n_{r,\infty}}$ we have

$$\begin{aligned}
\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P)z_{\mathbf{j}}| &= \sum_{k=1}^{n} \sum_{\mathbf{j}\in\mathcal{J}(m-1,k)} \left| c_{(\mathbf{j},k)}(P) w_{\mathbf{j}} w_k \frac{z_{\mathbf{j}}}{w_{\mathbf{j}}} \frac{z_k}{w_k} \right| \\
&\leqslant \sum_{k=1}^{n} \left|\frac{z_k}{w_k}\right| \left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,k)} |c_{(\mathbf{j},k)}(P_w)|^2 \right)^{\frac{1}{2}} \left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,k)} \left|\frac{z_{\mathbf{j}}}{w_{\mathbf{j}}}\right|^2 \right)^{\frac{1}{2}} \\
&\leqslant \sum_{k=1}^{n} \left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,k)} |c_{(\mathbf{j},k)}(P_w)|^2 \right)^{\frac{1}{2}} \sup_{k=1,\dots,n} \left|\frac{z_k}{w_k}\right| \left(\sum_{\mathbf{j}\in\mathcal{J}(m-1,k)} \left|\frac{z_{\mathbf{j}}}{w_{\mathbf{j}}}\right|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Finally, by Remark 6.3.1 we have

$$\left|\frac{z_k}{w_k}\right| \leqslant \|z\|_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)} \frac{\Phi^\delta(k)}{k^{1/\varphi(r)} \cdot w_k},$$

and replacing this bound in the previous chain of inequalities the proof is finished. □

Lemma 6.3.8 holds true for every $w \in B_{\ell^n_{r,s}}$. It will be convenient to pick w depending on r, s and $1/s < \delta < 1/2$ as follows

$$w = w_\delta = \begin{cases} \left(\frac{1}{k^{1/r}}\right)_{k=1}^{N} & \text{if } s = \infty \\ C_{r,s,\delta}\left(\frac{1}{k^{1/r}\log(k+1)^\delta}\right)_{k=1}^{N} & \text{if } s < \infty, \end{cases} \tag{6.13}$$

where $C_{r,s,\delta} > 0$ is such that $\|w\|_{\ell^n_{r,s}} \leqslant 1$. We may think that $C_{r,s,\delta} = 1$ when $s = \infty$.

Now we will need the *tetrahedral* and *even* multi-indices used in the *factorization decomposition* in Section 6.1.

Lemma 6.3.9. *Fix $2 \leqslant r \leqslant \infty$, $2 < s \leqslant \infty$ and $1/s < \delta < 1/2$. For every pair $M, N \in \mathbb{N}$, and every decreasing $z \in \mathbb{C}^N$ we have*

$$\sum_{\alpha\in\Lambda_T(M,N)} \left|\frac{z^\alpha}{w^\alpha}\right|^2 \leqslant C_{r,s,\delta}^{-2M}(1+\varepsilon)^M \|z\|^{2M}_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)} (N+1)^{\frac{1}{1+\varepsilon}},$$

for every $\varepsilon > 0$ and

$$\sum_{\alpha\in\Lambda_E(M,N)} \left|\frac{z^\alpha}{w^\alpha}\right|^2 \leqslant C_{r,s,\delta}^{-2M} \|z\|^{2M}_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)} \left(\sum_{k=1}^{\infty} \left(\frac{\log(k)}{k}\right)^2 \right)^{M/2},$$

where $w \in B_{\ell^N_{r,s}}$ and $C_{r,s,\delta}$ as defined in (6.13), .

Proof. We begin with the first inequality, observing that it is obvious if $N = 1$. We may, then, assume $N \geqslant 2$. Then, given $\alpha \in \Lambda_T(M,N)$, note that $\alpha! = 1$ and $|[\alpha]|$ is exactly $M!$. Then, for $s = \infty$ we have

$$\begin{aligned}\sum_{\alpha\in\Lambda_T(M,N)} \left|\frac{z^\alpha}{w^\alpha}\right|^2 &= \frac{1}{M!}\sum_{\alpha\in\Lambda_T(M,N)} \left|\frac{z^\alpha}{w^\alpha}\right|^2 M! \leqslant \frac{1}{M!}\left(\sum_{k=1}^N \left|\frac{z_k}{w_k}\right|^2\right)^M \\ &= \frac{1}{M!}\left(\|z\|^2_{\ell_{\varphi(r),2}}\right)^M \leqslant \frac{1}{M!}\|z\|^{2M}_{X_{\varphi(r),2}(\Psi_2)} \log(N+1)^M .\end{aligned} \tag{6.14}$$

On the other hand for $2 < s < \infty$ let us take $1/s < \delta < 1/2$. First by Remark 6.3.1 we have

$$\begin{aligned}\left|\frac{z_k}{w_k}\right|^2 &= C^{-2}_{r,s,\delta}\, |z_k|^{\varphi(s)}\, |z_k|^{2-\varphi(s)}\, k^{2/r} \log(k+1)^{2\delta} \\ &\leqslant C^{-2}_{r,s,\delta}|z_k|^{\varphi(s)}\|z\|^{2-\varphi(s)}_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)}\ \Phi^\delta(k)^{2-\varphi(s)}\ k^{\frac{\varphi(s)}{\varphi(r)}-1}\ \log(k+1)^{2\delta}.\end{aligned} \tag{6.15}$$

Then, proceeding as before, and using the expression in (6.15) it follows

$$\begin{aligned}\sum_{\alpha\in\Lambda_T(M,N)} \left|\frac{z^\alpha}{w^\alpha}\right|^2 &\leqslant \frac{1}{M!}\left(\sum_{k=1}^N \left|\frac{z_k}{w_k}\right|^2\right)^M \\ &\leqslant \frac{1}{M!}\left(C^{-2}_{r,s,\delta}\|z\|^{2-\varphi(s)}_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)}\Phi^\delta(N)^{2-\varphi(s)}\log(N+1)^{2\delta}\sum_{k=1}^N k^{\frac{\varphi(s)}{\varphi(r)}-1}|z_k|^{\varphi(s)}\right)^M \\ &= \frac{1}{M!}C^{-2M}_{r,s,\delta}\left(\|z\|^{2-\varphi(s)}_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)}\Phi^\delta(N)^{2-\varphi(s)}\log(N+1)^{2\delta}\|(z^*_k)_{k=1}^N\|^{\varphi(s)}_{\ell_{\varphi(r),\varphi(s)}}\right)^M \\ &= \frac{1}{M!}C^{-2M}_{r,s,\delta}\|z\|^{2M}_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)}\left(\Phi^\delta(N)^2\log(N+1)^{2\delta}\right)^M .\end{aligned} \tag{6.16}$$

Since $\Phi^\delta(N)^2\log(N+1)^{2\delta} = \log(N+1)$, and by the chain of inequalities in (6.16) we have

$$\sum_{\alpha\in\Lambda_T(M,N)} \left|\frac{z^\alpha}{w^\alpha}\right|^2 \leqslant \frac{C^{-2M}_{r,s,\delta}}{M!}\|z\|^{2M}_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)}\log(N+1)^M,$$

as for $s = \infty$.

A simple calculus argument shows that the function $f : [1,\infty[\to \mathbb{R}$ given by $f(x) = \frac{\log(x)^M}{x^{1/(1+\varepsilon)}}$ is bounded by $\left(\frac{(1+\varepsilon)M}{e}\right)^M$, then $\log(N)^M \leqslant N^{1/(1+\varepsilon)}\left(\frac{(1+\varepsilon)M}{e}\right)^M$. On the other hand $M! \geqslant \left(\frac{M}{e}\right)^M$. This gives the conclusion.

For the proof of the second inequality let us recall first that for each $\alpha \in \Lambda_E(M,N)$

there is a unique $\beta \in \Lambda(M/2, N)$ such that $\alpha = 2\beta$, then

$$\sum_{\alpha \in \Lambda_E(M,N)} \left| \frac{z^\alpha}{w^\alpha} \right|^2 = \sum_{\beta \in \Lambda(M/2,N)} \left| \frac{z^{2\beta}}{w^{2\beta}} \right|^2 = \sum_{\beta \in \Lambda(M/2,N)} \left| \frac{z^\beta}{w^\beta} \right|^4 \leqslant \left(\sum_{k=1} \left| \frac{z_k}{w_k} \right|^4 \right)^{M/2}$$
$$\leqslant C_{r,s,\delta}^{-2M} \|z\|^{2M}_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)} \left(\sum_{k=1}^{\infty} \left(\frac{\log(k+1)}{k} \right)^2 \right)^{M/2},$$

where we used Remark 6.3.1 and that it holds $\frac{\Phi^\delta(k)}{w_k} \leqslant C_{r,s,\delta}^{-1} \log(k+1)^{1/2} k^{1/r}$ for any $2 < s \leqslant \infty$ and every $k \in \mathbb{N}$. □

Lemma 6.3.10. *Fix $2 \leqslant r \leqslant \infty$, $2 < s \leqslant \infty$ and $2 < \delta < s$. For every pair $M, N \in \mathbb{N}$, and every decreasing $z \in \mathbb{C}^N$ we have*

$$\sum_{\alpha \in \Lambda(M,N)} \left| \frac{z^\alpha}{w^\alpha} \right|^2 \leqslant (M+1)(1+\varepsilon)^M A^M \|z\|^{2M}_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)} (N+1)^{\frac{1}{1+\varepsilon}},$$

for every $\varepsilon > 0$ and $A = C_{r,s,\delta}^{-2} \left(\sum_{l=1}^{\infty} \left(\frac{\log(l+1)}{l} \right)^2 \right)^{1/2}$.

Proof. Choose some decreasing z and use Lemma 6.3.9 to get

$$\begin{aligned}
\sum_{\alpha \in \Lambda(M,N)} \left| \frac{z^\alpha}{w^\alpha} \right|^2 &\leqslant \sum_{k=0}^{M} \sum_{\alpha_T \in \Lambda_T(k,N)} \sum_{\alpha_E \in \Lambda_E(M-k,N)} \left| \frac{z^{(\alpha_T+\alpha_E)}}{w^{(\alpha_T+\alpha_E)}} \right|^2 \\
&\leqslant \sum_{k=0}^{M} \left(\sum_{\alpha_T \in \Lambda_T(k,N)} \left| \frac{z^{\alpha_T}}{w^{\alpha_T}} \right|^2 \right) \left(\sum_{\alpha_E \in \Lambda_E(M-k,N)} \left| \frac{z^{\alpha_E}}{w^{\alpha_E}} \right|^2 \right) \\
&\leqslant \sum_{k=0}^{M} \left(C_{r,s,\delta}^{-2k} (1+\varepsilon)^k \|z\|^{2k}_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)} (N+1)^{\frac{1}{1+\varepsilon}} \right) \left(\|z\|^{2(M-k)}_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)} A^{M-k} \right) \\
&= \|z\|^{2M}_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)} (N+1)^{\frac{1}{1+\varepsilon}} A^M \sum_{k=0}^{M} (1+\varepsilon)^k \\
&\leqslant (M+1)(1+\varepsilon)^M A^M \|z\|^{2M}_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)} (N+1)^{\frac{1}{1+\varepsilon}}.
\end{aligned}$$
□

We are finally in the position to give the proof of Theorem 6.2.3 from which (as we already saw) the lower inclusion in Theorem 6.2.1 follows.

Proof of Theorem 6.3.7. Fix $2 \leqslant r \leqslant \infty$ and $2 < s \leqslant \infty$ and let $1/s < \delta < 1/2$ and $\varepsilon > 0$. For $n, m \in \mathbb{N}$ take $P \in \mathcal{P}(^m\mathbb{C}^n)$ and $z \in \mathbb{C}^n$. Since $\|z\|_{m\Psi_r} = \|z^*\|_{m\Psi_r}$, we may assume

$z = z^*$. Using Lemma 6.3.10 with $M = m-1$, $N = k$ and w as in (6.13) we have

$$\begin{aligned}
&\sup_{k=1,\dots,n} \frac{\Phi^\delta(k)}{k^{1/\varphi(r)} \cdot w_k} \left(\sum_{\mathbf{j} \in \mathcal{J}(m-1,k)} \left| \frac{z_\mathbf{j}}{w_\mathbf{j}} \right|^2 \right)^{\frac{1}{2}} \\
&\leqslant \sup_{k=1,\dots,n} \frac{\log(k+1)^{\frac{1}{2}-\delta} \log(k+1)^\delta k^{1/r}}{k^{1/\varphi(r)}} \left(m(1+\varepsilon)^{m-1} A^{m-1} \|z\|^{2(m-1)}_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)} k^{\frac{1}{1+\varepsilon}} \right)^{1/2} \\
&\leqslant \sqrt{m(1+\varepsilon)^{m-1} A^{m-1}} \|z\|^{(m-1)}_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)} \sup_{k=1,\dots,n} \frac{\log(k+1)^{1/2} k^{\frac{1}{2(1+\varepsilon)}}}{k^{1/2}}.
\end{aligned} \tag{6.17}$$

Since the sequence $\left(\frac{\log(k+1)^{1/2} k^{\frac{1}{2(1+\varepsilon)}}}{k^{1/2}} \right)_{k \geqslant 1}$ is eventually decreasing it follows

$$\sup_{k=1,\dots,n} \frac{\log(k+1)^{1/2} k^{\frac{1}{2(1+\varepsilon)}}}{k^{1/2}} = D(\varepsilon),$$

and replacing this in the chain of inequalities in (6.17) we obtain

$$\sup_{k=1,\dots,n} \frac{\Phi^\delta(k)}{k^{1/\varphi(r)} \cdot w_k} \left(\sum_{\mathbf{j} \in \mathcal{J}(m-1,k)} \left| \frac{z_\mathbf{j}}{w_\mathbf{j}} \right|^2 \right)^{\frac{1}{2}} \leqslant \sqrt{m(1+\varepsilon)^{m-1} A^{m-1}} D(\varepsilon) \|z\|^{(m-1)}_{X_{\varphi(r),\varphi(s)}(\Phi^\delta)} \tag{6.18}$$

Lemma 6.3.8 plus the bound in (6.18) yield

$$\begin{aligned}
\sum_{\mathbf{j} \in \mathcal{J}(m,n)} |c_\mathbf{j}(P) z_\mathbf{j}| &\leqslant em 2^{m-1} \|P\|_{\mathcal{P}(^m\ell^n_{r,\infty})} \|z\|_{X_{\varphi(r),2}} \sup_{k=1,\dots,n} \left(\frac{\log(k)}{k} \sum_{\mathbf{j} \in \mathcal{J}(m-1,k)} \left| \frac{z_\mathbf{j}}{w_\mathbf{j}} \right|^2 \right)^{\frac{1}{2}} \\
&\leqslant em^{3/2} 2^{m-1} C(\varepsilon) \sqrt{m(1+\varepsilon)^{m-1} A^{m-1}} \|P\|_{\mathcal{P}(^m\ell^n_{r,\infty})} \|z\|^m_{X_{\varphi(r),2}}.
\end{aligned}$$

where A is defined as in Lemma 6.3.10. □

It is remarkable that in those Banach sequence spaces X for which we were able to characterize the set of monomial convergence for the family $H_b(X)$ it results that $monH_b(X)$ is itself a Banach sequence space. This brings the new questions.

Question 6.3.11. *Is the set of monomial convergence of $H_b(X)$ a Banach sequence space for any given Banach sequence space X?*

Or the less ambitious one.

Question 6.3.12. *Is it always a Banach space or at least a vector space?*

Those questions seem very interesting and they trace a research path around the structure of these sets.

Chapter 7

Monomial convergence for $H_\infty(B_{\ell_r})$

We will use the results in Chapter 6 to shed new light on the sets of monomial convergence of $H_\infty(B_{\ell_{r,s}})$. Previously our methods allowed us to characterize the resulting Banach sequence space $monH_b(\ell_r)$ whenever $1 < r \leqslant 2$. Here we transpose those results to give new descriptions of $monH_\infty(B_{\ell_r})$ when $1 < r \leqslant 2$.

7.1 Changing finite coordinates

When dealing with $monH_\infty(B_{\ell_\infty})$ it is very useful the fact that, if a sequence belongs to the set of monomial convergence and we modify finitely many coordinates, the resulting sequence remains in the set of monomial convergence.

Lemma 7.1.1. *[DGMPG08, Lemma 2] If $z \in H_\infty(B_{\ell_\infty})$ and $u \in B_{\ell_\infty}$ satisfy that $|u_n| \leqslant |z_n|$ for all but finitely many $n \in \mathbb{N}$, then $u \in monH_\infty(B_{\ell_\infty})$.*

It is unknown whether or not an analogous result holds for ℓ_r (see the comments regarding this problem in [Sch15, Chapter 10]). We overcome this with the following proposition, which is a weaker version of this, but enough for our purposes. We are inspired by [DGMPG08, Lemma 2] and [DGMSP19, Proposition 10.14].

Proposition 7.1.2. *Let $1 < r < \infty$ and $u, z \in B_{\ell_r}$ be such that $|u_n| \leqslant |z_n|$ for $1 \leqslant n \leqslant N$ and $|u_n| = |z_n|$ for $n > N$. Suppose that there exists $\rho > \sum_{n=1}^N |z_n|^r$ so that $u \in monH_\infty((1-\rho)^{1/r} B_{\ell_r})$. Then $z \in monH_\infty(B_{\ell_r})$.*

Proof. Let $a_1, \ldots, a_N$ be positive real numbers such that $|z_i| < a_i$ for every $1 \leqslant i \leqslant N$ and

$$a := \sum_{n=1}^N a_n^r < \rho.$$

Given $f \in H_\infty(B_{\ell_r})$ and $k_1, \ldots, k_N \in \mathbb{N}$, we define (following the proof of [DGMPG08, Lemma 2])

$$f_{k_1,\ldots,k_N}(\nu) := \frac{1}{(2\pi i)^N} \int_{|w_1|=a_1} \cdots \int_{|w_N|=a_N} \frac{f(w_1, \ldots, w_N, \nu_{N+1}, \nu_{N+2}, \ldots)}{w_1^{k_1+1} \cdots w_N^{k_N+1}} dw_1 \cdots dw_N.$$

Note that $f_{k_1,\ldots,k_N}$ is well defined on the contracted ball $(1-a)^{1/r}B_{\ell_r}$ and, in fact, belongs to $H_\infty((1-a)^{1/r}B_{\ell_r})$ (because $f \in H_\infty(B_{\ell_r})$) and

$$\|f_{k_1,\ldots,k_N}\|_{(1-a)^{1/r}\cdot B_{\ell_r}} \leqslant \frac{\|f\|_{B_{\ell_r}}}{a_1^{k_1}\cdots a_N^{k_N}}. \tag{7.1}$$

Our next step is to understand the coefficients $c_\alpha(f_{k_1,\ldots,k_N})$ in relation to those of f. For each multi-index $\alpha = (\alpha_1,\ldots,\alpha_n,0,\ldots)$ with $\alpha_n \neq 0$, an application of the Cauchy integral formula yields

$$c_\alpha(f_{k_1,\ldots,k_n}) = \begin{cases} c_{(k_1,\ldots,k_N,\alpha_{N+1},\ldots,\alpha_n)}(f) & \text{if } \alpha_1 = \cdots = \alpha_N = 0,\\ 0 & \text{otherwise.}\end{cases} \tag{7.2}$$

We have now everything we need to proceed. Note that, since $a < \rho$, we have $u \in monH_\infty((1-\rho)^{1/r}B_{\ell_r}) \subset monH_\infty((1-a)^{1/r}B_{\ell_r})$. With Proposition 3.1.4 and (7.1) we get

$$\sum_{\beta\in\mathbb{N}_0^{(\mathbb{N})}} |c_\beta(f_{k_1,\ldots,k_N})||u_{N+1}^{\beta_1}\cdots u_{N+2}^{\beta_2}\cdots| \leqslant C_u\|f_{k_1,\ldots,k_N}\|_{(1-a)^{1/r}B_{\ell_r}} \leqslant C_u\frac{\|f\|_{B_{\ell_r}}}{a_1^{k_1}\cdots a_N^{k_N}}. \tag{7.3}$$

Now using (7.2) and (7.3) (recall that $|u_n| = |z_n|$ for $n \geqslant N+1$) we have

$$\begin{aligned}
\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f)||z^\alpha| &= \sum_{(k_1,\ldots k_N)\in\mathbb{N}_0^N} |z_1^{k_1}\cdots z_N^{k_N}| \sum_{\beta\in\mathbb{N}_0^{(\mathbb{N})}} |c_{(k_1,\ldots,k_N,\beta)}(f)||u_{N+1}^{\beta_1}\cdots u_{N+2}^{\beta_2}\cdots|\\
&= \sum_{(k_1,\ldots k_N)\in\mathbb{N}_0^N} |z_1^{k_1}\cdots z_N^{k_N}| \sum_{\beta\in\mathbb{N}_0^{(\mathbb{N})}} |c_\beta(f_{k_1,\ldots,k_N})||u_{N+1}^{\beta_1}\cdots u_{N+2}^{\beta_2}\cdots|\\
&\leqslant \sum_{(k_1,\ldots k_N)\in\mathbb{N}_0^N} |z_1^{k_1}\cdots z_N^{k_N}| C_u\frac{\|f\|_{B_{\ell_r}}}{a_1^{k_1}\cdots a_N^{k_N}}\\
&= C_u\|f\|_{B_{\ell_r}}\prod_{n=1}^N\sum_{k_n\geqslant 0}\left(\frac{|z_n|}{a_n}\right)^{k_n} < \infty,
\end{aligned}$$

as we wanted. □

Let us make the last observation before we proceed with the following sections. Given a Banach sequence space X, for every $f \in H_\infty(tB_X)$ and $t > 0$ the function f_t given by $f_t(x) = f(tx)$ for $x \in B_X$ belongs to $H_\infty(B_X)$ and $c_\alpha(f_t) = t^{|\alpha|}c_\alpha(f)$ for every α. Then, if $z \in monH_\infty(B_X)$ we have

$$\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f)(tz)^\alpha| = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f)t^{|\alpha|}z^\alpha| = \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f_t)z^\alpha| < \infty.$$

This implies $t\, monH_\infty(B_X) \subset monH_\infty(tB_X)$ for every Banach sequence space X and every $t > 0$.

Noting that tB_X is the open unit ball of the Banach sequence space $(X, t\|\cdot\|_X)$, the previous inclusion yields

$$t^{-1}monH_\infty(tB_X) \subset monH_\infty(t^{-1}tB_X) = monH_\infty(B_X).$$

This altogether shows

$$monH_\infty(tB_X) = t\,monH_\infty(B_X) \tag{7.4}$$

for every Banach sequence space X and every $t > 0$.

7.2 The case $1 \leqslant r \leqslant 2$

Now we will focus on $monH_\infty(B_{\ell_r})$ for $1 < r \leqslant 2$. The main result of the section is the following theorem that, in some sense that will become clear later (see Remark **??**), characterizes the geometry of the set of monomial convergence for this families of holomorphic functions.

Theorem 7.2.1. *Let $1 < r \leqslant 2$ then,*

$$\left\{z \in B_{\ell_r} : 2e\|id : m_{\Psi_r} \to \ell_r\|^r \left(\limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}}\right)^r + \|z\|_{\ell_r}^r < 1\right\} \subset$$
$$monH_\infty(B_{\ell_r}) \subset \left\{z \in B_{\ell_r} : \limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} \leqslant 1\right\}.$$

The upper inclusion follows using probabilistic techniques, as in the case of $monH_b(\ell_r)$. The lower inclusion, on the other hand, relies on Theorem 6.2.3 and requires some preliminary work that starts with the following lema.

Lemma 7.2.2. *Let $1 < r \leqslant 2$ then, $\frac{1}{\|id:m_{\Psi_r}\to\ell_r\|(2e)^{1/r}} B_{m_{\Psi_r}} \subset monH_\infty(B_{\ell_r})$.*

Proof. In order to keep things readable we write $K = \|id : m_{\Psi_r} \to \ell_r\|(2e)^{1/r}$. We first show that if $z \in \frac{1}{K} B_{m_{\Psi_r}}$ is non-decreasing, then $z \in monH_\infty(B_{\ell_r})$. The general result follows from the fact that $B_{m_{\Psi_r}}$ and $monH_\infty(B_{\ell_r})$ are both symmetric (Corollary 3.3.6). We choose now $f \in H_\infty(B_{\ell_r})$ and fix $\varepsilon > 0$ so that $(1+\varepsilon)^{1/r}\|z\|_{m_{\Psi_r}} K < 1$. By Theorem 6.2.3 we can find $C_r(\varepsilon) > 0$ so that

$$\begin{aligned}
\sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |c_\alpha(f) z^\alpha| &= \sup_{n\in\mathbb{N}} \sum_{m=0}^\infty \sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(f) z_{\mathbf{j}}| \\
&\leqslant \sup_{n\in\mathbb{N}} \sum_{m=0}^\infty C_r(\varepsilon) m^{2+\frac{1}{r}} (1+\varepsilon)^{\frac{m}{r}} K^m \|z\|_{m_{\Psi_r}}^m \sup_{u\in B_{\ell_r^n}} \left| \sum_{\mathbf{j}\in\mathcal{J}(m,n)} c_{\mathbf{j}}(f) u_{\mathbf{j}} \right| \\
&\leqslant \sum_{m=0}^\infty C_r(\varepsilon) \left(m^{\frac{1}{m}(2+\frac{1}{r})} (1+\varepsilon)^{\frac{1}{r}} K \|z\|_{m_{\Psi_r}}\right)^m \|P_m(f)\|_{\mathcal{P}(^m\ell_r)} \\
&\leqslant \|f\|_{B_{\ell_r}} C_r(\varepsilon) \sum_{m=0}^\infty \left(m^{\frac{1}{m}(2+\frac{1}{r})} (1+\varepsilon)^{\frac{1}{r}} K \|z\|_{m_{\Psi_r}}\right)^m.
\end{aligned}$$

The choice of ε and fact that $m^{\frac{1}{m}(2+\frac{1}{r})} \to 1$ as $m \to \infty$ immediately gives that the series converges and completes the proof. □

We are now in conditions to prove Theorem 7.2.1.

Proof of Theorem 7.2.1. Let us start with the upper inclusion

$$monH_\infty(B_{\ell_r}) \subset \Big\{ z \in B_{\ell_r} \colon \limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} \leqslant 1 \Big\}.$$

Fix $z \in monH_\infty(B_{\ell_r})$. Arguing as in the proof of the upper inclusion of Theorem 6.2.1, proceeding as in (6.2), replacing the role of Lemma 6.2.2 by Proposition 3.1.4, and as in (6.3) we get

$$\sum_{j=1}^n |z_j^*| \leqslant C_{z^*,r}^{\frac{1}{m}} \Big[\log(m)^{\frac{1}{m}} (2\pi m)^{\frac{1}{2m}} e^{\frac{1}{12m^2}} \frac{m}{e} n^{\frac{1}{m}} \Big]^{1-\frac{1}{r}}.$$

where $C_{z^*,r}$ is a positive constant that depends only on z^* and r. Choosing $m = \lfloor \log(n+1) \rfloor$ we get

$$\limsup_{n\to\infty} \frac{1}{\log(n+1)^{1-\frac{1}{r}}} \sum_{k=1}^n |z_n^*| \leqslant 1,$$

which gives our claim.
We now face the proof of the lower inclusion

$$\Big\{ z \in \mathbb{C}^{\mathbb{N}} \colon 2e\|id : m_{\Psi_r} \to \ell_r\|^r \Big(\limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} \Big)^r + \|z\|_{\ell_r}^r < 1 \Big\} \subset monH_\infty(B_{\ell_r}).$$

In order to keep the notation as simple as possible, let $K = 2e\|id : m_{\Psi_r} \to \ell_r\|^r$. Take $z \in \mathbb{C}^{\mathbb{N}}$ such that

$$K \Big(\limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} \Big)^r + \|z\|_{\ell_r}^r < 1,$$

and note that this implies $z \in B_{\ell_r}$. Denote $L := \limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}}$, choose $\varepsilon > 0$ so that

$$K\big((1+\varepsilon)L\big)^r + \|z\|_{\ell_r}^r < 1, \tag{7.5}$$

and $N \in \mathbb{N}$ for which

$$\sup_{n \geqslant N} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} < (1+\varepsilon)L.$$

Let us observe that

$$z_N^* < \frac{\log(N+1)^{1-1/r}}{N}(1+\varepsilon)L, \tag{7.6}$$

(this follows essentially as in Remark 6.2.4) and define $u = (\underbrace{z_N^*, \ldots, z_N^*}_{N}, z_{N+1}^*, z_{N+2}^*, \ldots)$.

First, for every $n < N$ we have, using (7.6),

$$\frac{\sum_{k=1}^n u_k^*}{\log(n+1)^{1-1/r}} < (1+\varepsilon)L.$$

On the other hand, for $n \geqslant N$,

$$\frac{\sum_{k=1}^n u_k^*}{\log(n+1)^{1-1/r}} \leqslant \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} < (1+\varepsilon)L.$$

This altogether gives $\|u\|_{m_{\Psi_r}} < (1+\varepsilon)L$. We choose $\rho > \sum_{k=1}^N |z_k|^r$ such

$$\|id : m_{\Psi_r} \to \ell_r\|^r (2e)(L(1+\varepsilon))^r + \rho < 1,$$

and, using (7.5) we get

$$\|u\|_{m_{\Psi_r}} < (1+\varepsilon)L < \frac{(1-\rho)^{1/r}}{\|id : m_{\Psi_r} \to \ell_r\|(2e)^{1/r}}.$$

Lemma 7.2.2 and equation (7.4) imply $u \in monH_\infty((1-\rho)^{1/r}B_{\ell_r})$ and, then Proposition 7.1.2 gives $z^* \in monH_\infty(B_{\ell_r})$. Finally, Corollary 3.3.6 yields $z \in monH_\infty(B_{\ell_r})$ and completes the proof. □

Remark 7.2.3. Theorem 7.2.1 implies other known results which try to characterize the set of monomial convergence of $H_\infty(B_{\ell_r})$. Note first that, if $z \in \ell_1$, then

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} = 0.$$

Thus

$$B_{\ell_r} \cap \ell_1 \subset \left\{ z \in \mathbb{C}^{\mathbb{N}} : 2e\|id : m_{\Psi_r} \to \ell_r\|^r \left(\limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} \right)^r + \|z\|_{\ell_r}^r < 1 \right\}.$$

On the other hand, if $z \in B_{\ell_r}$ is such that

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} \leqslant 1$$

then there is a constant $c > 0$ so that

$$z_n^* \leqslant c \frac{\log(n+1)^{1-1/r}}{n}.$$

From this we easily get that $z \in \ell_{1+\varepsilon}$ for every $\varepsilon > 0$, and we recover a result of [DMP09] (see (3.8)).

The following corollary extends the result in (3.10) enlarging the range of values for θ from $(1/2, \infty)$ to $(0, \infty)$. This solves an explicit question given in [BDS19, Remark 5.9]. Even more, the result enables us to take $\theta = 0$ multiplying the sequence by a constant. This means that, in some sense, Theorem 7.2.1 actually gives a better understanding of $monH_\infty(B_{\ell_r})$ for $1 < r \leqslant 2$.

Corollary 7.2.4. *Let $1 < r \leqslant 2$, then, for every $\theta > 0$, it holds*

$$\left(\frac{1}{n^{1/r'}\log(n+2)^{\theta}}\right)_{n\geqslant 1} \cdot B_{\ell_r} \subset monH_\infty(B_{\ell_r}). \tag{7.7}$$

Moreover, denoting $K = \frac{1}{(2e\|id:m_{\Psi_r}\to\ell_r\|+1)^{1/r}}$, we have

$$\left(\frac{1}{Kn^{1/r'}}\right)_{n\geqslant 1} \cdot B_{\ell_r} \subset monH_\infty(B_{\ell_r}). \tag{7.8}$$

Proof. Let us begin by proving (7.7). Fix $\theta > 0$ and choose $z \in \left(\frac{1}{n^{1/r'}\log(n+2)^{\theta}}\right)_{n\geqslant 1} B_{\ell_r}$. We can find $w \in B_{\ell_r}$ so that $z_n = \frac{w_n}{n^{1/r'}\log(n+1)^{\theta}}$ for every $n \in \mathbb{N}$. Since $z \in c_0$, there is an injective $\sigma : \mathbb{N} \to \mathbb{N}$ such that $z_n^* = |z_{\sigma(n)}| = \frac{|w_{\sigma(n)}|}{\sigma(n)^{1/r'}\log(\sigma(n)+2)^{\theta}}$. Using Hölder's inequality we get

$$\begin{aligned}
\frac{1}{\log(n+1)^{1/r'}}\sum_{l=1}^{n} z_l^* &= \frac{1}{\log(n+1)^{1/r'}}\sum_{l=1}^{n}\frac{|w_{\sigma(l)}|}{\sigma(l)^{1/r'}\log(\sigma(l)+2)^{\theta}} \\
&\leqslant \frac{1}{\log(n+1)^{1/r'}}\left(\sum_{l=1}^{n}|w_{\sigma(l)}|^r\right)^{1/r}\left(\sum_{l=1}^{n}\frac{1}{\sigma(l)\log(\sigma(l)+2)^{r'\theta}}\right)^{1/r'} \\
&\leqslant \frac{1}{\log(n+1)^{1/r'}}\left(\sum_{l=1}^{n}\frac{1}{\sigma(l)\log(\sigma(l)+2)^{r'\theta}}\right)^{1/r'} \\
&\leqslant \frac{1}{\log(n+1)^{1/r'}}\left(\sum_{l=1}^{n}\frac{1}{l\log(l+2)^{r'\theta}}\right)^{1/r'},
\end{aligned}$$

where the last inequality holds because $x \mapsto \frac{1}{x\log(x+2)^{r'\theta}}$ defines a decreasing function for $x > 1$. The last term, $\frac{1}{\log(n+1)^{1/r'}}\left(\sum_{l=1}^{n}\frac{1}{l\log(l+2)^{r'\theta}}\right)^{1/r'}$, goes to 0 as $n \to \infty$, and therefore

$$\limsup_{n\to\infty}\frac{1}{\log(n+1)^{1/r'}}\sum_{l=1}^{n} z_l^* = 0.$$

Indeed, suppose that $\theta < \frac{1}{r'}$ (which me may always asume since $\frac{1}{l\log(l+2)^{r'\theta}}$ is decreasing on θ). Thus, there is some $C_{r',\theta} > 0$ such that

$$\begin{aligned}
\left(\sum_{l=1}^{n}\frac{1}{l\log(l+2)^{r'\theta}}\right)^{1/r'} &\leqslant C_{r',\theta}\left(\int_{l=2}^{n}\frac{1}{x\log(x)^{r'\theta}}dx\right)^{1/r'} \\
&= C_{r',\theta}\left(\int_{l=\log(2)}^{\log(n)}\frac{1}{y^{r'\theta}}dy\right)^{1/r'} \\
&\leqslant C_{r',\theta}\log(n)^{-\theta+\frac{1}{r'}},
\end{aligned}$$

Then, $\frac{1}{\log(n+1)^{1/r'}} \left(\sum_{l=1}^n \frac{1}{l \log(l+2)^{r'\theta}} \right)^{1/r'} \leqslant C_{r',\theta} \log(n)^{-\theta} \to 0$.

On the other hand, $z \in B_{\ell_r}$ (note that $|z_n| \leqslant |w_n|$ for every n and $w \in B_{\ell_r}$), then

$$2e\|id : m_{\Psi_r} \to \ell_r\|^r \left(\limsup_{n\to\infty} \frac{\sum_{k=1}^n z_l^*}{\log(n+1)^{1-1/r}} \right)^r + \|z\|_{\ell_r}^r = \|z\|_{\ell_r}^r < 1,$$

and, by Theorem 7.2.1, $z \in monH_\infty(B_{\ell_r})$.

To prove (7.8), take $z = \left(\frac{1}{Kn^{1/r'}} w_n \right)_{n\geqslant 1}$ with $w \in B_{\ell_r}$, and note that $\|z\|_{\ell_r}^r < \frac{1}{K^r}$. Since $z \in c_0$, there is an injective $\sigma : \mathbb{N} \to \mathbb{N}$ such that $z_n^* = |z_{\sigma(n)}| = \frac{|w_{\sigma(n)}|}{K\sigma(n)^{1/r'}}$. Using Hölder's inequality we get

$$\begin{aligned} \frac{K}{\log(n+1)^{1/r'}} \sum_{l=1}^n z_l^* &= \frac{1}{\log(n+1)^{1/r'}} \sum_{l=1}^n \frac{|w_{\sigma(l)}|}{\sigma(l)^{1/r'}} \\ &\leqslant \frac{1}{\log(n+1)^{1/r'}} \left(\sum_{l=1}^n |w_{\sigma(l)}|^r \right)^{1/r} \left(\sum_{l=1}^n \frac{1}{\sigma(l)} \right)^{1/r'} \\ &\leqslant \frac{1}{\log(n+1)^{1/r'}} \left(\sum_{l=1}^n \frac{1}{\sigma(l)} \right)^{1/r'} \\ &\leqslant \frac{1}{\log(n+1)^{1/r'}} \left(\sum_{l=1}^n \frac{1}{l} \right)^{1/r'} \leqslant 1. \end{aligned}$$

Since $K = (2e\|id : m_{\Psi_r} \to \ell_r\|^r + 1)^{1/r}$, we have

$$2e\|id : m_{\Psi_r} \to \ell_r\|^r \left(\limsup_{n\to\infty} \frac{\sum_{k=1}^n z_l^*}{\log(n+1)^{1-1/r}} \right)^r + \|z\|_{\ell_r}^r < (2e\|id : m_{\Psi_r} \to \ell_r\|^r + 1)\frac{1}{K^r} = 1.$$

Now Theorem 7.2.1 gives the conclusion. □

Let us analyze Theorem 7.2.1 in a qualitative way.

Observe that, given $1 < r \leqslant 2$, we may define the following two norms

$$\|z\|_{A_r} = \max\left\{ \|z\|_{\ell_r}, \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} \right\}, \tag{7.9}$$

$$\|z\|_{\tilde{A}_r} = K_r \left(\limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} \right)^r + \|z\|_{\ell_r}^r \tag{7.10}$$

for every $z \in m_{\Psi_r}$ where $K_r = 2e\|id : m_{\Psi_r} \to \ell_r\|^r$. Let us call A_r and $\tilde{A}_r$ to the spaces defined through these norms respectively. Recall the well known bound

$$\max\{|a|, |b|\} \leqslant (|a|^r + |b|^r)^{1/r} \leqslant 2^{\frac{1}{r}} \max\{|a|, |b|\}, \tag{7.11}$$

which holds for every $1 \leqslant r < \infty$ and all $a, b \in \mathbb{C}$. Thanks to (7.11), the norms defined in (7.9) and (7.10) are equivalent. Thus, for some constant $C_r > 0$, we have

$$C_r \cdot B_{A_r} \subset B_{\tilde{A}_r}. \tag{7.12}$$

Note that

$$B_{A_r} = \Big\{z \in B_{\ell_r} \colon \limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}} < 1\Big\},$$
$$B_{\tilde{A}_r} = \Big\{z \in B_{\ell_r} \colon K_r \left(\limsup_{n\to\infty} \frac{\sum_{k=1}^n z_k^*}{\log(n+1)^{1-1/r}}\right)^r + \|z\|_{\ell_r}^r < 1\Big\},$$

and thanks to Theorem 7.2.1 and the set inclusion in (7.12) it follows that

$$C_r \cdot B_{A_r} \subset monH_\infty(B_{\ell_r}) \subset \overline{B_{A_r}}. \tag{7.13}$$

Remark 7.2.5. If we hope a result as in equation (6.6), i.e.,

$$B_r \subset monH_\infty(B_{\ell_r}) \subset \overline{B_r},$$

with B_r the ball of some normed space then, the bounds in (7.13) imply that, its norm needs to be equivalent to the norm in A_r. In this sense Theorem 7.2.1 characterizes the geometry of $monH_\infty(B_{\ell_r})$.

To end this chapter we will apply some of the results in this section to give a new way to tackle a problem we have already addressed in Chapter 5.

7.3 Mixed Bohr radius revisited

In this last section we present an application of the results given in the chapter to the mixed Bohr radius. As a consequence of Lemma 6.2.5 we can give an alternative proof of the lower bounds for $K(B_{\ell_p^n}, B_{\ell_q^n})$ for the case $1 \leqslant p \leqslant 2$ (and every $1 \leqslant q \leqslant \infty$). We will show now a fact that we have already proved in Section 5.5.3, this is for $1 < q < p \leqslant 2$

$$\frac{\log(n)^{1-1/p}}{n^{1-1/q}} \ll K(B_{\ell_p^n}, B_{\ell_q^n}).$$

By Theorem 4.2.2 and Lemma 7.2.2, there is a constant $C := C(p) > 0$ such that for every polynomial P in n complex variables we have

$$\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}| \leqslant C^m \|z\|_{(m_{\Psi_p})_n}^m \|P\|_{\mathcal{P}(^m\ell_p^n)}, \tag{7.14}$$

where $(m_{\Psi_p})_n$ is defined as the quotient space induced by the mapping

$$\begin{aligned} \pi_n : m_{\Psi_p} &\to \mathbb{C}^n \\ x &\mapsto (x_1, \dots, x_n). \end{aligned}$$

Note that there is a constant $D = D(p,q) > 0$ such that $\|z\|_{(m_{\Psi_p})_n} \leqslant D \frac{n^{1-\frac{1}{q}}}{\log(n)^{1-\frac{1}{p}}} \|z\|_{\ell_q^n}$. Therefore, by (7.14) we have

$$\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}| \leqslant (CD)^m \left(\frac{n^{1-\frac{1}{q}}}{\log(n)^{1-\frac{1}{p}}}\right)^m \|z\|_{\ell_q^n}^m \|P\|_{\mathcal{P}(^m\ell_p^n)},$$

This implies that $\chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n))^{1/m} \ll \frac{n^{1-\frac{1}{q}}}{\log(n)^{1-\frac{1}{p}}}$. It should be noted that here it is important to have a control of the growth of the (p,q)-mixed unconditional constant also in terms of m (the homogeneity degree), in fact we need the hypercontractivity given in (7.14).The result now follows using Lemma 5.3.2, i.e.,

$$K(B_{\ell_p^n}, B_{\ell_q^n}) \sim \frac{1}{\sup_{m\geqslant 1} \chi_{p,q}(\mathcal{P}(^m\mathbb{C}^n))^{1/m}}.$$

Chapter 8

Monomial convergence for $\mathcal{P}(^m\ell_r)$

In this final chapter we revisit the set of monomial convergence of homogeneous polynomials. For some fixed $1 < r \leqslant 2$ and $m \geqslant 2$ we have proved in Section 3.4 that

$$\ell_q \subset mon\mathcal{P}(^m\ell_r),$$

for $q = (mr')'$ answering an open question. Our aim now is to tighten this lower bound. We find a lower inclusion that gets narrower when m gets bigger.

Theorem 8.0.1. *Fix $1 < r \leqslant 2$ and, for each $m \geqslant 2$, define $q := (mr')'$. Then $\ell_q \subset mon\mathcal{P}(^2\ell_r)$; $\ell_{q,2} \subset mon\mathcal{P}(^3\ell_r)$; $\ell_{q,\frac{3+\sqrt{5}}{2}} \subset mon\mathcal{P}(^4\ell_r)$ and*

$$\ell_{q,\frac{m}{\log(m)}} \subset mon\mathcal{P}(^m\ell_r).$$

for $m \geqslant 5$.

Our starting point will be the results proved in Section 3.4. In this way, Corollary 3.4.2 proves the case $m = 2$ in Theorem 8.0.1. We face now the problem of getting the result for other m's. The general philosophy is always to try to get a bound as that in Theorem 3.4.1. There, in the right-hand-side we have some constants that depend on r and m (but not on n, the number of variables), the norm of the polynomial and the norm of z in some space X. This then implies $X \subset mon\mathcal{P}(^m\ell_r)$. Henceforth the idea is to take the sum as depending on m different variables; that is, for each polynomial P we consider

$$\sum_{1\leqslant j_1\leqslant\cdots\leqslant j_m\leqslant n} |c_{\mathbf{j}}(P) z^{(1)}_{j_1} \dots z^{(m)}_{j_m}| \tag{8.1}$$

with $z^{(1)}, \dots, z^{(m)} \in \mathbb{C}^n$ and then try to get an estimate that involves the norms of the $z^{(j)}$ in (possibly) different spaces. This then gives that the smallest of these spaces is contained in the set of monomial convergence (see Remark 8.2.2). We do this (giving the proof of Theorem 8.0.1) in two stages (that we present in the following two sections). First we give an estimate for the sum that involves both $\ell_{q,1}$ and $\ell_{q,\infty}$ norms (the precise statement is given in Proposition 8.1.1). Then we interpret this inequality as operators from $\ell_{q,\infty}\times\cdots\times\ell_{q,\infty}\times\ell_{q,1}\times\ell_{q,\infty}\times\cdots\times\ell_{q,\infty}$ to $\ell_1(\mathcal{J}(m,n))$ and use interpolation techniques to improve the $\ell_{q,1}$-norm (by weakening the $\ell_{q,\infty}$-norm). This is done in Theorem 8.2.1. What happens here is that, since in the estimate in Proposition 8.1.1 some of the variables have to be decreasing, we cannot use general multilinear interpolation, but interpolation in cones instead (a more detailed explanation is given in Section 8.2).

8.1 First bound for the sum

As we announced, our first step towards the proof of Theorem 8.0.1 is to get a bound for a sum like that in (8.1). This becomes the main result of this section.

Proposition 8.1.1. *Let $1 < r \leqslant 2$ and $m \geqslant 2$. Define $q := (mr')'$. There exists $C_{m,r} > 1$ so that for every $n \in \mathbb{N}$, every $P \in \mathcal{P}(^m\mathbb{C}^n)$, every $z^{(1)},\ldots,z^{(m)} \in \mathbb{C}^n$ and $1 \leqslant k \leqslant m-1$ we have*

$$\sum_{1\leqslant j_1\leqslant\cdots\leqslant j_m\leqslant n} \big|c_{\mathbf{j}}(P) z^{(1)}_{j_1}\cdots z^{(k)}_{j_k} z^{(k+1)*}_{j_{k+1}}\cdots z^{(m)*}_{j_m}\big| \leqslant C_{m,r}\|z^{(k)}\|_{\ell_{q,1}}\prod_{i\neq k}\|z^{(i)}\|_{\ell_{q,\infty}}\|P\|_{\mathcal{P}(^m\ell_r^n)}\,.$$

The proof requires some work, that we prepare with a few lemmas. But before let us make a couple of elementary comments. First of all, by definition,

$$z^*_k \leqslant \|z\|_{\ell_{q,\infty}}\frac{1}{k^{1/q}} \tag{8.2}$$

for every $z \in \mathbb{C}^n$ and, then

$$\sum_{k=N}^{M} z^*_k \leqslant \|z\|_{\ell_{q,\infty}}\sum_{k=N}^{M}\frac{1}{k^{1/q}}\,. \tag{8.3}$$

Also, for $-1 \neq \alpha < 0$,

$$\sum_{k=N}^{M} n^\alpha = N^\alpha + \sum_{k=N+1}^{M} n^\alpha \leqslant N^\alpha + \int_N^M x^\alpha dx = N^\alpha + \frac{1}{\alpha+1}\big(M^{\alpha+1}-N^{\alpha+1}\big)\,. \tag{8.4}$$

Lemma 8.1.2. *Let $n,k \geqslant 1$ and $1 \leqslant q < \infty$. Then for every $z^{(1)},\ldots,z^{(k)} \in \mathbb{C}^n$ and $1 \leqslant j \leqslant n$ we have*

$$\sum_{1\leqslant j_1\leqslant\cdots\leqslant j_k\leqslant j} |z^{(1)}_{j_1}\ldots z^{(k)}_{j_k}| \leqslant (q')^k j^{\frac{k}{q'}}\prod_{1\leqslant i\leqslant k}\|z^{(i)}\|_{\ell_{q,\infty}}.$$

Proof. We proceed by induction on k. For $k=1$ the statement is a straightforward consequence of (8.3) and (8.4). Assume that the result holds for $k-1$. Then

$$\begin{aligned}\sum_{1\leqslant j_1\leqslant\cdots\leqslant j_k\leqslant j} |z^{(1)}_{j_1}\cdots z^{(k)}_{j_l}| &= \sum_{j_k=1}^{j}|z^{(k)}_{j_k}|\Big(\sum_{1\leqslant j_1\leqslant\cdots\leqslant j_{k-1}\leqslant j_k}|z^{(1)}_{j_1}\ldots z^{(k-1)}_{j_{k-1}}|\Big)\\ &\leqslant (q')^{k-1}\prod_{1\leqslant i\leqslant k-1}\|z^{(i)}\|_{\ell_{q,\infty}} j_k^{\frac{k-1}{q'}}\sum_{j_k=1}^{j}|z^{(k)}_{j_k}| \leqslant (q')^k j^{\frac{k-1}{q'}} j^{\frac{1}{q'}}\prod_{1\leqslant i\leqslant k}\|z^{(i)}\|_{\ell_{q,\infty}}\,,\end{aligned}$$

which concludes the proof. ∎

Lemma 8.1.3. *Let $1 < r \leqslant 2$, $m \geqslant 3$ and $n \in \mathbb{N}$. Fix $q := (mr')'$ and $1 \leqslant k \leqslant m-2$. For every $z^{(i_1)}, \dots, z^{(i_k)} \in \mathbb{C}^n$ and $1 \leqslant t \leqslant n$ we have*

$$\sum_{t \leqslant j_1 \leqslant \dots \leqslant j_k \leqslant n} |z_{j_1}^{(i_1)*} \dots z_{j_k}^{(i_k)*}| j_k^{\frac{1}{r}-\frac{1}{q}} \leqslant \Big(\prod_{1 \leqslant l \leqslant k} \big(\frac{mr'}{m-l-1} + \frac{1}{t} \big) \Big) t^{\frac{k+1}{q'}-\frac{1}{r'}} \Big(\prod_{1 \leqslant l \leqslant k} \| z^{(i_l)} \|_{\ell_{q,\infty}} \Big).$$

Proof. First of all let us note that a simple computation shows that $\frac{s}{q'} - \frac{1}{r'} \leqslant -\frac{1}{mr'} < 0$ for every $1 \leqslant s \leqslant m-1$. We now proceed by induction on k. For $k = 1$ we use (8.3) and (8.4) to have

$$\begin{aligned}
\sum_{j=t}^{n} |z_j^*| j^{\frac{1}{r}-\frac{1}{q}} &\leqslant \|z\|_{\ell_{q,\infty}} \sum_{j=t}^{n} j^{\frac{2}{q'}-\frac{1}{r'}-1} \\
&\leqslant \|z\|_{\ell_{q,\infty}} \Big(t^{\frac{2}{q'}-\frac{1}{r'}} - \big(\frac{2}{q'} - \frac{1}{r'}\big)^{-1} t^{\frac{2}{q'}-\frac{1}{r'}} \Big) = \big(\frac{r'm}{m-2} + \frac{1}{t} \big) t^{\frac{2}{q'}-\frac{1}{r'}} \|z\|_{\ell_{q,\infty}}.
\end{aligned}$$

Let us suppose now that the statement holds for $k-1$ and prove it for k.

$$\begin{aligned}
&\sum_{t \leqslant j_1 \leqslant \dots \leqslant j_k \leqslant n} |z_{j_1}^{(i_1)*} \dots z_{j_k}^{(i_k)*}| j_k^{\frac{1}{r}-\frac{1}{q}} \\
&= \sum_{j_1=t}^{n} |z_{j_1}^{(i_1)*}| \sum_{j_1 \leqslant j_2 \leqslant \dots \leqslant j_k \leqslant n} |z_{j_2}^{(i_2)*} \dots z_{j_k}^{(i_k)*}| j_k^{\frac{1}{r}-\frac{1}{q}} \\
&\leqslant \sum_{j_1=t}^{n} |z_{j_1}^{(i_1)*}| \Big(\prod_{1 \leqslant l \leqslant k-1} \big(\frac{mr'}{m-l-1} + \frac{1}{j_1} \big) \Big) j_1^{\frac{k}{q'}-\frac{1}{r'}} \Big(\prod_{2 \leqslant l \leqslant k} \| z^{(i_l)} \|_{\ell_{q,\infty}} \Big) \\
&\leqslant \Big(\prod_{1 \leqslant l \leqslant k-1} \big(\frac{mr'}{m-l-1} + \frac{1}{t} \big) \Big) \Big(\prod_{2 \leqslant l \leqslant k} \| z^{(i_l)} \|_{\ell_{q,\infty}} \Big) \sum_{j_1=t}^{n} |z_{j_1}^{(i_1)*}| j_1^{\frac{k}{q'}-\frac{1}{r'}} \\
&\leqslant \Big(\prod_{1 \leqslant l \leqslant k-1} \big(\frac{mr'}{m-l-1} + \frac{1}{t} \big) \Big) \Big(\prod_{1 \leqslant l \leqslant k} \| z^{(i_l)} \|_{\ell_{q,\infty}} \Big) \sum_{j_1=t}^{n} j_1^{\frac{k+1}{q'}-\frac{1}{r'}-1} \\
&\leqslant \Big(\prod_{1 \leqslant l \leqslant k-1} \big(\frac{mr'}{m-l-1} + \frac{1}{t} \big) \Big) \Big(\prod_{1 \leqslant l \leqslant k} \| z^{(i_l)} \|_{\ell_{q,\infty}} \Big) t^{\frac{k+1}{q'}-\frac{1}{r'}} \Big(\frac{1}{t} - \big(\frac{k+1}{q'} - \frac{1}{r'} \big)^{-1} \Big) \\
&= \Big(\prod_{1 \leqslant l \leqslant k-1} \big(\frac{mr'}{m-l-1} + \frac{1}{t} \big) \Big) \Big(\prod_{1 \leqslant l \leqslant k} \| z^{(i_l)} \|_{\ell_{q,\infty}} \Big) t^{\frac{k+1}{q'}-\frac{1}{r'}} \Big(\frac{1}{t} + \frac{mr'}{m-k-1} \Big). \qquad \square
\end{aligned}$$

Lemma 8.1.4. *Let $1 < r \leqslant 2$, $m \geqslant 3$. Fix $q := (mr')'$ and $1 \leqslant k \leqslant m-2$. For every $z^{(1)}, \dots, z^{(k)} \in \mathbb{C}^n$ we have*

$$\sum_{1 \leqslant j_1 \leqslant \dots \leqslant j_{m-1} \leqslant n} |z_{j_1}^{(1)} \dots z_{j_k}^{(k)} z_{j_{k+1}}^{(k+1)*} \dots z_{j_{m-1}}^{(m-1)*}| j_{m-1}^{\frac{1}{r}-\frac{1}{q}} \leqslant (q'+1)^{m-2} \|z^{(k)}\|_{\ell_{q,1}} \prod_{\substack{1 \leqslant i \leqslant m-1 \\ i \neq k}} \|z^{(i)}\|_{\ell_{q,\infty}}.$$

Proof. We begin by splitting the sum in a convenient way

$$\sum_{1\leqslant j_1\leqslant\cdots\leqslant j_{m-1}\leqslant n} |z^{(1)}_{j_1}\cdots z^{(k)}_{j_k} z^{(k+1)*}_{j_{k+1}}\cdots z^{(m-1)*}_{j_{m-1}}| j_{m-1}^{\frac{1}{r}-\frac{1}{q}}$$
$$= \sum_{j_k=1}^{n} |z^{(k)}_{j_k}| \Big(\sum_{j_k\leqslant j_{k+1}\leqslant\cdots\leqslant j_{m-1}\leqslant n} |z^{(k+1)*}_{j_{k+1}}\cdots z^{(m-1)*}_{j_{m-1}}| j_{m-1}^{\frac{1}{r}-\frac{1}{q}}\Big)\Big(\sum_{1\leqslant j_1\leqslant\cdots\leqslant j_{k-1}\leqslant j_k} |z^{(1)}_{j_1}\cdots z^{(k-1)}_{j_{k-1}}|\Big).$$

We fix j_k and bound the first block using Lemma 8.1.3, taking into account that we have now $m-k-1$ z's and that $\frac{1}{j_k}+\frac{mr'}{m-l-1}\leqslant q'+1$ for every $1\leqslant l\leqslant m-k-1$,

$$\sum_{j_k\leqslant j_{k+1}\leqslant\cdots\leqslant j_{m-1}\leqslant n} |z^{(k+1)*}_{j_{k+1}}\cdots z^{(m-1)*}_{j_{m-1}}| j_{m-1}^{\frac{1}{r}-\frac{1}{q}}$$
$$\leqslant j_k^{\frac{m-k}{q'}-\frac{1}{r'}} \Big(\prod_{1\leqslant l\leqslant m-k-1} \frac{1}{j_k}+\frac{mr'}{m-l-1}\Big)\Big(\prod_{k+1\leqslant i\leqslant m-1} \|z^{(i)}\|_{\ell_{q,\infty}}\Big)$$
$$\leqslant j_k^{\frac{m-k}{q'}-\frac{1}{r'}} (q'+1)^{m-k-1} \prod_{k+1\leqslant i\leqslant m-1} \|z^{(i)}\|_{\ell_{q,\infty}}.$$

With this, and bounding the second block using Lemma 8.1.2 we get

$$\sum_{1\leqslant j_1\leqslant\cdots\leqslant j_{m-1}\leqslant n} |z^{(1)}_{j_1}\cdots z^{(k)}_{j_k} z^{(k+1)*}_{j_{k+1}}\cdots z^{(m-1)*}_{j_{m-1}}| j_{m-1}^{\frac{1}{r}-\frac{1}{q}}$$
$$\leqslant (q'+1)^{m-2}\prod_{i\neq k}\|z^{(i)}\|_{\ell_{q,\infty}} \sum_{j_k=1}^{n} |z^{(k)}_{j_k}| j_k^{\frac{k-1}{q'}+\frac{m-k}{q'}-\frac{1}{r'}}.$$

It easy to see that $\frac{k-1}{q'}+\frac{m-k}{q'}-\frac{1}{r'}=\frac{1}{q}-1$. Therefore, using *Hardy-Littlelwood rearrangement inequality* in Lemma 6.3.5 we have

$$\sum_{j_k=1}^{n} |z^{(k)}_{j_k}| j_k^{\frac{1}{q}-1} \leqslant \sum_{j_k=1}^{n} |(z^{(k)})^*_{j_k}| j_k^{\frac{1}{q}-1} = \|z^{(k)}\|_{\ell_{q,1}}. \qquad \square$$

As it was the case for the study of holomorphic functions, Theorem 2.1.7 (in fact (2.9)) is a crucial tool for the proof of Proposition 8.1.1.

Proof of Proposition 8.1.1. We begin by using Hölder's inequality and (2.9) (noting that

$|\mathbf{i}| \leqslant (m-1)!$ for every $\mathbf{i} \in \mathcal{J}(m-1,n)$) and (8.2) to have

$$\begin{aligned}
&\sum_{\mathbf{j}\in\mathcal{J}(m,n)} \big|c_{\mathbf{j}}(P) z^{(1)}_{j_1}\cdots z^{(k)}_{j_k} z^{(k+1)*}_{j_{k+1}} \cdots z^{(m)*}_{j_m}\big| \\
&\quad = \sum_{1\leqslant j_1 \leqslant \cdots \leqslant j_{m-1}\leqslant n} |z^{(1)}_{j_1} \ldots z^{(m-1)*}_{j_{m-1}}| \sum_{j_m=j_{m-1}}^{n} |c_{\mathbf{j}}(P) z^{(m)*}_{j_m}| \\
&\quad \leqslant \sum_{\mathbf{j}\in\mathcal{J}(m-1,n)} |z^{(1)}_{j_1} \ldots z^{(m-1)*}_{j_{m-1}}| \Big(\sum_{j_m=j_{m-1}}^{n} c_{\mathbf{j}}(P)^{r'} \Big)^{\frac{1}{r'}} \Big(\sum_{j_m=j_{m-1}}^{n} |z^{(m)*}_{j_m}|^r \Big)^{\frac{1}{r}} \\
&\quad \leqslant (m-1)!^{\frac{1}{r}} m e^{1+\frac{m-1}{r}} \|P\|_{\mathcal{P}(^m\ell_r^n)} \|z^{(m)}\|_{\ell_{q,\infty}} \sum_{\mathbf{j}\in\mathcal{J}(m-1,n)} |z^{(1)}_{j_1} \ldots z^{(m-1)*}_{j_{m-1}}| \Big(\sum_{j_m=j_{m-1}}^{n} j_m^{-\frac{r}{q}} \Big)^{\frac{1}{r}}.
\end{aligned}$$

Observe now that, for each $N \in \mathbb{N}$ we have $N^{-r/q} \leqslant 2^{r/q} x^{-r/q}$ for every $N \leqslant x < N+1$. Then

$$\sum_{j_m=j_{m-1}}^{n} j_m^{-\frac{r}{q}} \leqslant 2^{\frac{r}{q}} \int_{j_{m-1}}^{n} x^{-\frac{r}{q}} dx \leqslant 2^{\frac{r}{q}} \frac{q}{r-q} j_{m-1}^{1-\frac{r}{q}}.$$

The proof now finishes with a straightforward application of Lemma 8.1.4. □

8.2 Real interpolation on cones.

Now we are going to look at summability inequalities for polynomials from the point of view of its associated multilinear mappings. We fix a polynomial $P \in \mathcal{P}(^m\mathbb{C}^n)$ and consider the mapping $\mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \ell_1(\mathcal{J}(m,n))$, given by

$$(z^{(1)}, \ldots, z^{(m)}) \mapsto \big(c_{\mathbf{j}}(P) z^{(1)}_{j_1} \ldots z^{(m)}_{j_m}\big)_{\mathbf{j}\in\mathcal{J}(m,n)}. \tag{8.5}$$

Note that, since everything here is finite dimensional, the mapping is well defined. The idea is, then, to consider norms on the domain spaces so that the norm of this mapping is bounded by a term involving the norm of the polynomial and some constant independent of n. Since the inequality that we get in Proposition 8.1.1 requires some variables to be decreasing we have to restrict ourselves to cones of decreasing sequences. To be more precise, if we denote $\ell^d_{q,s} := \{z \in \ell_{q,s} \colon |z| = z^*\}$ for $1 \leqslant q, s \leqslant \infty$, Proposition 8.1.1 tells us that there is a constant $C_{m,r} > 1$ (independent of P and n) such that, for every $1 \leqslant k \leqslant m-1$, the mapping

$$T_k : \underbrace{\ell^n_{q,\infty} \times \cdots \times \ell^n_{q,\infty}}_{k-1} \times \ell^n_{q,1} \times \underbrace{(\ell^n_{q,\infty})^d \times \cdots \times (\ell^n_{q,\infty})^d}_{m-k} \to \ell_1(\mathcal{J}(m,n)), \tag{8.6}$$

given by (8.5) satisfies

$$\|T_k\| \leqslant C_{m,r} \|P\|_{\mathcal{P}(^m\ell^n_r)}. \tag{8.7}$$

All these mappings have the same defining formula (which is m-linear), so it is tempting to apply multilinear interpolation. But, since we need to restrict ourselves to the cone of non-increasing sequences in the last $m-k$ variables, we are not able to directly apply the classical multilinear interpolation results, and we will have to apply interpolation on cones. For the general theory of interpolation we follow (and refer the reader to) [BL76]. As far as we know there is no theory of interpolation of multilinear mappings definded on cones of normed spaces. Since (as we have already explained) we have to consider linear operators on cones, we use the K-method of interpolation for operators on the cone of non-increasing sequences, as presented in [CM96]. Then the main result of this section, from which Theorem 8.0.1, follows is the following.

Theorem 8.2.1. *Let $1 < r \leqslant 2$ and $m \geqslant 3$. Define $q := (mr')'$ and*

$$s = \begin{cases} 2 & \text{if } m = 3 \\ \frac{3+\sqrt{5}}{2} & \text{if } m = 4 \\ \frac{m}{\log(m)} & \text{if } m \geqslant 5 \end{cases}$$

There exists a constant $C_{m,r} \geqslant 1$ such that, for every $P \in \mathcal{P}(^m\mathbb{C}^n)$ the m-linear mapping

$$T : \underbrace{(\ell^n_{q,s})^d \times \cdots \times (\ell^n_{q,s})^d}_{m-1} \times (\ell^n_{q,\infty})^d \to \ell_1(\mathcal{J}(m,n))$$

given by

$$(z^{(1)}, \ldots, z^{(m)}) \mapsto \big(c_{\mathbf{j}}(P) z^{(1)}_{j_1} \ldots z^{(m)}_{j_m}\big)_{\mathbf{j} \in \mathcal{J}(m,n)}$$

satisfies

$$\|T\| \leqslant C_{m,r} \|P\|_{\mathcal{P}(^m\ell^n_r)} .$$

Remark 8.2.2. If we take $z^{(1)} = \ldots = z^{(m)} = z$ and observe that $\|z\|_{\ell_{q,\infty}} \leqslant \|z\|_{\ell_{q,s}}$, Theorem 8.2.1 gives

$$\sum_{1 \leqslant j_1 \leqslant \cdots \leqslant j_m \leqslant n} |c_{\mathbf{j}}(P) z^*_{j_1} \cdots z^*_{j_m}| \leqslant C_{m,r} \|z\|^m_{\ell_{q,s}} \|P\|_{\mathcal{P}(^m\ell^n_r)}$$

for every $P \in \mathcal{P}(^m\mathbb{C}^n)$ and $z \in \mathbb{C}^n$. A standard argument shows that $z^* \in mon\mathcal{P}(^m\ell_r)$ for every $z \in \ell_{q,s}$ and, then, Corollary 3.3.6 implies $\ell_{q,s} \subset mon\mathcal{P}(^m\ell_r)$. This gives Theorem 8.0.1.

Before we proceed, let us fix some notation. Given a Banach function lattice X (in particular a sequence space or a finite dimensional Banach space, on which we are mainly interested), we write X^d for the *cone of non-increasing functions in X*. If Y is any Banach space and $S : X \to Y$ is a linear operator we can restrict it to the cone and denote

$$\|S : X^d \to Y\| = \inf\{\|S(x)\|_Y : x \in X^d,\ \|x\| < 1\} . \tag{8.8}$$

Clearly neither is X^d a vector space, nor is $\|S\|$ a norm. We will later use an analogous notation for m-linear mappings. We are now ready to state our main tool to interpolate in cones. It is a direct corollary of [CM96, Theorem 1–(b)] (recall that we are using the notation as introduced there).

Theorem 8.2.3. *Given a pair of quasi-Banach function lattices (X_0, X_1), a pair of quasi-Banach spaces (Y_0, Y_1) and a linear operator S defined both $X_0 \to Y_0$ and $X_1 \to Y_1$ with*

$$\|S : X_0^d \longrightarrow Y_0\| \leqslant M_0 \quad \text{and} \quad \|S : X_1^d \longrightarrow Y_1\| \leqslant M_1 .$$

Then for every $0 < \theta < 1$ the operator $S : (X_0^d, X_1^d)_{\theta,a} \longrightarrow (Y_0, Y_1)_{\theta,a}$ is well defined and

$$\|S : (X_0^d, X_1^d)_{\theta,a} \longrightarrow (Y_0, Y_1)_{\theta,a}\| \leqslant M_0^{1-\theta} M_1^{\theta}.$$

We are going to apply this to Lorentz sequence spaces. In this case, it was proved in [Sag72] (see also [CM96, Theorem 4]) that

$$(\ell_{q,p_0}^d, \ell_{q,p_1}^d)_{\theta,a} = (\ell_{q,p_0}, \ell_{q,p_1})_{\theta,a}^d .$$

On the other hand, it is known (see for example [BL76, Theorem 5.3.1]) that whenever $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ we have

$$(\ell_{q,p_0}, \ell_{q,p_1})_{\theta,p} = \ell_{q,p},$$

and therefore

$$(\ell_{q,p_0}^d, \ell_{q,p_1}^d)_{\theta,p} = \ell_{q,p}^d. \tag{8.9}$$

Finally [BL76, Theorem 3.7.1] gives that (if p_0, p_1, p are related as before)

$$(\ell'_{q,p_0}, \ell'_{q,p_1})_{\theta,p} = (\ell_{q,p_0}, \ell_{q,p_1})'_{\theta,p} = \ell'_{q,p} . \tag{8.10}$$

The idea now is to use Theorem 8.2.3 to interpolate multilinear mappings. Let us explain how we are going to do this. Let $X_1, \dots, X_m$ be Banach function lattices (in our case they will always be finite dimensional Lorentz spaces), Y some Banach space ($\ell_1(\mathcal{J}(m,n)$ for us) and some continuous m-linear $T : X_1 \times \cdots \times X_m \to Y$ (for us given by (8.5)). Now we fix $1 \leqslant j \neq k \leqslant m$ and, for each $i \neq j, k$ pick $z^{(i)} \in X_i$ and $\varphi \in Y'$ and consider $v = (z^{(1)}, \dots, z^{(m)}, \varphi)$. Now we define

$$T_v : X_j \to X'_k \quad \text{by} \quad \big(T_v(z^{(j)})\big)(z^{(k)}) = \varphi(T(z^{(1)}, \dots, z^{(m)})). \tag{8.11}$$

An easy computation shows that

$$\|T_v\| \leqslant \|\varphi\| \, \|T\| \prod_{i \neq j,k} \|z^{(i)}\| , \tag{8.12}$$

and that

$$\|T\| = \sup_{\varphi \in B_{Y'}, z^{(i)} \in B_{X_i}} \|T_v\|. \tag{8.13}$$

Observe that in this procedure we may consider X_i^d for every i except for $i = k$, getting the same estimate for the norm (defining the "norm" for multilinear mappings on cones with the same idea as in (8.8)). We are now ready to present the main technical tool for the proof of Theorem 8.2.1.

Lemma 8.2.4. *Let $m \geqslant 3$, $1 < r \leqslant 2$, define $q := (mr')'$ and let $C_{m,r}$ be the constant from Proposition 8.1.1. For each $0 < \theta < 1$, every $P \in \mathcal{P}(^m\mathbb{C}^n)$ and all $1 \leqslant k \leqslant m-2$ the m-linear mapping*

$$T^k(\theta) : \big(\ell^n_{q,(\frac{1}{1-\theta})^k}\big)^d \times \underbrace{\big(\ell^n_{q,\frac{1}{\theta}}\big)^d \times \cdots \times \big(\ell^n_{q,\frac{1}{\theta}}\big)^d}_{k} \times \underbrace{\big(\ell^n_{q,\infty}\big)^d \times \cdots \times \big(\ell^n_{q,\infty}\big)^d}_{m-k-1} \to \ell_1(\mathcal{J}(m,n))$$

given by (8.5) *satisfies*

$$\|T^k(\theta)\| \leqslant C_{m,r}\|P\|_{\mathcal{P}(^m\ell_r^n)}.$$

Proof. We proceed by induction on k and begin with the case $k=1$. We consider the mappings (see (8.6))

$$T_1 : \ell^n_{q,1} \times \underbrace{(\ell^n_{q,\infty})^d \times (\ell^n_{q,\infty})^d \times \cdots \times (\ell^n_{q,\infty})^d}_{m-1} \to \ell_1(\mathcal{J}(m,n))$$

$$T_2 : \ell^n_{q,\infty} \times \ell^n_{q,1} \times \underbrace{(\ell^n_{q,\infty})^d \times \cdots \times (\ell^n_{q,\infty})^d}_{m-2} \to \ell_1(\mathcal{J}(m,n)).$$

We fix $z^{(3)},\dots,z^{(m)} \in (\ell^n_\infty)^d$ and $\varphi \in \Big(\ell_1(\mathcal{J}(m,n))\Big)'$ and writing $v = (z^{(3)},\dots,z^{(m)},\varphi)$ define, following (8.11), two linear operators

$$(T_1)_v : \big(\ell^n_{q,\infty}\big)^d \to \big(\ell^n_{q,1}\big)' \quad \text{and} \quad (T_2)_v : \big(\ell^n_{q,1}\big)^d \to \big(\ell^n_{q,\infty}\big)'$$

that, by (8.7) and (8.12), satisfy (for $i=1,2$)

$$\|(T_i)_v\| \leqslant C_{m,r}\|P\|_{\mathcal{P}(^m\ell_r^n)}\|z^{(3)}\|_{\ell_{q,\infty}}\cdots\|z^{(m)}\|_{\ell_{q,\infty}}\|\varphi\|_{\ell_1(\mathcal{J}(m,n))'}.$$

Now we interpolate, using Theorem 8.2.3 and equations (8.9) and (8.10), to have

$$\Big\|\big(T^1(\theta)\big)_v : \big(\ell^n_{q,\frac{1}{\theta}}\big)^d \to \big(\ell^n_{q,\frac{1}{1-\theta}}\big)'\Big\| \leqslant C_{m,r}\|P\|_{\mathcal{P}(^m\ell_r^n)}\|z^{(3)}\|_{\ell_{q,\infty}}\cdots\|z^{(m)}\|_{\ell_{q,\infty}}\|\varphi\|_{\ell_1(\mathcal{J}(m,n))'}$$

for every $0<\theta<1$. Using equation (8.13) and taking supremum, this immediately gives

$$\|T^1(\theta) : \ell^n_{q,\frac{1}{1-\theta}} \times \big(\ell^n_{q,\frac{1}{\theta}}\big)^d \times \underbrace{\big(\ell^n_{q,\infty}\big)^d \times \cdots \times \big(\ell^n_{q,\infty}\big)^d}_{m-2} \to \ell_1(\mathcal{J}(m,n))\| \leqslant C_{m,r}\|P\|_{\mathcal{P}(^m\ell_r^n)}.$$

Now let us assume that, for $1 \leqslant k \leqslant m-2$,

$$T^{k-1}(\theta) : \ell^n_{q,\left(\frac{1}{1-\theta}\right)^{k-1}} \times \underbrace{(\ell^n_{q,\frac{1}{\theta}})^d \times \cdots \times (\ell^n_{q,\frac{1}{\theta}})^d}_{k-1} \times \underbrace{(\ell^n_{q,\infty})^d \times \cdots \times (\ell^n_{q,\infty})^d}_{m-k} \to \ell_1(\mathcal{J}(m,n))$$

has norm $\leqslant C_{m,r}\|P\|_{\mathcal{P}(^m\ell_r^n)}$. On the other hand consider the mapping defined by Theorem 8.1.1 (see (8.6))

$$T_{k+1} : \underbrace{\ell^n_{q,\infty} \times \cdots \times \ell^n_{q,\infty}}_{k} \times \ell^n_{q,1} \times \underbrace{(\ell^n_{q,\infty})^d \times \cdots \times (\ell^n_{q,\infty})^d}_{m-k-1} \to \ell_1(\mathcal{J}(m,n))$$

that (recall (8.12)) also has norm $\leqslant C_{m,r}\|P\|_{\mathcal{P}(^m\ell_r^n)}$. Since $\|\ell^n_{q,\frac{1}{\theta}} \hookrightarrow \ell^n_{q,\infty}\| = 1$ we have (recall (8.8))

$$T_{k+1} : \ell^n_{q,\infty} \times \underbrace{(\ell^n_{q,\frac{1}{\theta}})^d \times \cdots \times (\ell^n_{q,\frac{1}{\theta}})^d}_{k-1} \times \ell_{q,1} \times \underbrace{(\ell^n_{q,\infty})^d \times \cdots \times (\ell^n_{q,\infty})^d}_{m-k-1} \to \ell_1(\mathcal{J}(m,n))$$

has again norm bounded by $C_{m,r}\|P\|_{\mathcal{P}(^m\ell_r^n)}$. We fix $\varphi \in \big(\ell_1(\mathcal{J}(m,n))\big)'$ and $z^{(i)} \in (\mathbb{C}^n)^d$ for $i \neq 1, k$ and, taking $v = (z^{(2)}, \ldots, z^{(k)}, z^{(k+2)}, \ldots, z^{(m)}, \varphi)$ we have, by (8.11) and (8.12)

$$\begin{aligned}\big\|(T^{k-1}(\theta))_v : (\ell^n_{q,\infty})^d \to \big(\ell^n_{q,(\frac{1}{1-\theta})^{k-1}}\big)'\big\| \\ \leqslant C_{m,r}\|P\|_{\mathcal{P}(^m\ell_r^n)}\|\varphi\|_{\ell_1(\mathcal{J}(m,n))'}\|z^{(2)}\|_{\ell_{q,\frac{1}{\theta}}} \cdots \|z^{(k)}\|_{\ell_{q,\frac{1}{\theta}}}\|z^{(k+2)}\|_{\ell_{q,\infty}} \cdots \|z^{(m)}\|_{\ell_{q,\infty}}\end{aligned}$$

and

$$\begin{aligned}\big\|(T_{k+1})_v : (\ell^n_{q,1})^d \to \big(\ell^n_{q,\infty}\big)'\big\| \\ \leqslant C_{m,r}\|P\|_{\mathcal{P}(^m\ell_r^n)}\|\varphi\|_{\ell_1(\mathcal{J}(m,n))'}\|z^{(2)}\|_{\ell_{q,\frac{1}{\theta}}} \cdots \|z^{(k)}\|_{\ell_{q,\frac{1}{\theta}}}\|z^{(k+2)}\|_{\ell_{q,\infty}} \cdots \|z^{(m)}\|_{\ell_{q,\infty}}\,.\end{aligned}$$

Once again, we may interpolate using Theorem 8.2.3, (8.9) and (8.10) to have

$$\begin{aligned}\big\|(T^{k}(\theta)_v : (\ell^n_{q,\frac{1}{\theta}})^d \to \big(\ell^n_{q,\frac{1}{(1-\theta)^k}}\big)'\big\| \\ \leqslant C_{m,r}\|P\|_{\mathcal{P}(^m\ell_r^n)}\|\varphi\|_{\ell_1(\mathcal{J}(m,n))'}\|z^{(2)}\|_{\ell_{q,\frac{1}{\theta}}} \cdots \|z^{(k)}\|_{\ell_{q,\frac{1}{\theta}}}\|z^{(k+2)}\|_{\ell_{q,\infty}} \cdots \|z^{(m)}\|_{\ell_{q,\infty}}\end{aligned}$$

for every $0 < \theta < 1$. Taking supremum as before this gives that the norm of the multilinear form $T^k(\theta)$

$$\big\|T^k(\theta) : \ell^n_{q,\left(\frac{1}{1-\theta}\right)^k} \times \underbrace{(\ell^n_{q,\frac{1}{\theta}})^d \times \cdots \times (\ell^n_{q,\frac{1}{\theta}})^d}_{k} \times \underbrace{(\ell^n_{q,\infty})^d \times \cdots \times (\ell^n_{q,\infty})^d}_{m-k-1} \to \ell_1(\mathcal{J}(m,n))\big\|,$$

is bounded by $\leqslant C_{m,r}\|P\|_{\mathcal{P}(^m\ell_r^n)}$ for every $0 < \theta < 1$. □

Proof of Theorem 8.2.1. For $m \geqslant 5$, we choose $\theta = \frac{\log(m+\frac{3}{2})}{m-1+\log(m+\frac{3}{2})}$ and $k = m-2$. Then $\frac{1}{\theta} \geqslant \frac{m}{\log(m)}$ and

$$\left(\frac{1}{1-\theta}\right)^k = \Big(1 + \frac{\log(m+\frac{3}{2})}{m-1}\Big)^{m-2} \geqslant \frac{m}{\log m}.$$

Therefore $\|\ell^n_{q,\left(\frac{1}{1-\theta}\right)^k} \hookrightarrow \ell^n_{q,\frac{m}{\log(m)}}\| = \|\ell^n_{q,\frac{1}{\theta}} \hookrightarrow \ell^n_{q,\frac{m}{\log(m)}}\| = 1$. Using Lemma 8.2.4 with $k = m-2$ the result follows. For $m = 3$ take $\theta = \frac{1}{2}$ and $k = 1$ in Lemma 8.2.4, for $m = 4$ take $\theta = \frac{3}{2} - \frac{\sqrt{5}}{2}$ and $k = 2$. □

We finish this section with some comments on the hypercontractivity of the inclusion of $\ell_{q,s}$ in $mon\mathcal{P}(^m\ell_r)$. For the ℓ_∞ case it is known (see [BDF+17, Theorem 2.1]) that the inclusion $\ell_{\frac{2m}{m-1},\infty}$ in $mon\mathcal{P}(^m\ell_\infty)$ is hypercontractive in the sense that there exists a constant $C > 0$ such for every $P \in \mathcal{P}(^m\ell_\infty)$,

$$\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P)z_{\mathbf{j}}| \leqslant C^m \|z\|^m_{\ell_{\frac{2m}{m-1},\infty}} \|P\|_{P(^m\ell_\infty)}.$$

Remark 8.2.5. For $1 < r \leqslant 2$, although we do not know if $\ell_{q,\infty}$ lies in the set $mon\mathcal{P}(^m\ell_r)$ it is easy to see that we cannot expect to have a hypercontractive inequality as above.

If there exists such a constant, proceeding as in the proof of the upper inclusion in Theorem 6.2.1 (see (6.3)) with $m = \lfloor \log(n+1) \rfloor$ we would have that

$$\frac{1}{\|z\|_{\ell_{q,\log m}} \log(n+1)^{1-\frac{1}{r}}} \sum_{j=1}^n |z_j^*|$$

is bounded independently of n for every $z \in \ell_{q,\log m}$. Take now $z = (j^{-1/q}\log(j)^{-2/\log(m)})_j$. Then $\|z\|_{\ell_{q,\log m}} \leqslant \left(\sum_{j=1}^\infty \frac{1}{j\log^2(j)}\right)^{\frac{1}{\log m}}$. But,

$$\begin{aligned}\frac{1}{\|z\|_{\ell_{q,\log m}} \log(n+1)^{1-\frac{1}{r}}} \sum_{j=1}^n |z_j^*| &\gg \frac{1}{\log(n+1)^{1-\frac{1}{r}}} \sum_{j=1}^n \frac{1}{j^{1/q}\log(j)^{\frac{2}{\log m}}} \\ &\gg \frac{e^2}{c\log(n+1)^{1-\frac{1}{r}}} \sum_{j=1}^n \frac{1}{j^{1/q}} \geqslant \frac{e^2}{c\log(n+1)^{1-\frac{1}{r}}} n^{1/q'} q'.\end{aligned}$$

Since $q' = mr' = \lfloor \log(n+1) \rfloor r'$, the last expression is $\gg \log(n)^{\frac{1}{r}}$. This shows that there exists no constant $C > 0$ such that for every n and m and all $P \in \mathcal{P}(^m\mathbb{C}^n)$ we have

$$\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P)z_{\mathbf{j}}| \leqslant C^m \|z\|^m_{\ell_{q,\log m}} \|P\|_{\mathcal{P}(^m\ell_r^n)}.$$

On the other hand, applying carefully the ideas developed in this section, it is possible to obtain hypercontractive inequalities in some cases.

Remark 8.2.6. Given $\varepsilon > 0$, there exists a constant $C > 0$ such that for every $m \geqslant 3$, $n \in \mathbb{N}$ and every $P \in \mathcal{P}(^m\mathbb{C}^n)$

$$\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P)z_{\mathbf{j}}| \leqslant C(1+\varepsilon)^m \|P\|_{\mathcal{P}(^m\ell_r^n)} \|z\|^m_{\ell^n_{q,2}}.$$

To see this fix $1 < r \leqslant 2$, $m \geqslant 3$, and take z, $z^{(m-2)}$, $z^{(m-1)}$, $w \in \mathbb{C}^n$ such that $z^{(m-1)} =$

$z^{(m-1)*}$ and $w = w^*$. Then we have, using Theorem 2.1.7 and Lemma 3.4.3,

$$\begin{aligned}
&\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P) z_{j_1}\dots z_{j_{m-3}} z^{(m-2)}_{j_{m-2}} z^{(m-1)}_{j_{m-1}} w_{j_m}| \\
&\leqslant em\|P\|_{\mathcal{P}(^m\ell_r^n)} \sum_{\mathbf{j}\in\mathcal{J}(m-1,n)} |z_{j_1}\dots z_{j_{m-3}} z^{(m-2)}_{j_{m-2}} z^{(m-1)}_{j_{m-1}}| \cdot \Big(\frac{(m-1)^{m-1}}{\alpha(\mathbf{j})^{\alpha(\mathbf{j})}}\Big)^{1/r} \Big(\sum_{j_m=j_{m-1}}^{n} w_{j_m}^r\Big)^{1/r} \\
&\leqslant em^3 Cm^{e^{r'-1}} \|P\|_{\mathcal{P}(^m\ell_r^n)} \sum_{j_{m-2}=1}^{n} |z^{(m-2)}_{j_{m-2}}| \Big(\sum_{\mathbf{j}\in\mathcal{J}(m-3,j_{m-2})} |\mathbf{j}||z_{\mathbf{j}}|\Big) \sum_{j_{m-1}=j_{m-2}}^{n} |z^{(m-1)}_{j_{m-1}}| \Big(\sum_{j_m=j_{m-1}}^{n} w_{j_m}^r\Big)^{1/r} \\
&\leqslant Cm^{e^{r'}} \|w\|_{\ell_{q,\infty}} \|P\|_{\mathcal{P}(^m\ell_r^n)} \sum_{j_{m-2}=1}^{n} |z^{(m-2)}_{j_{m-2}}| \Big(\sum_{l=1}^{j_{m-2}} |z_l|\Big)^{m-3} \sum_{j_{m-1}=j_{m-2}}^{n} |z^{(m-1)}_{j_{m-1}}| j_{m-1}^{\frac{1}{r}-\frac{1}{q}} \\
&\leqslant Cm^{e^{r'}} \|w\|_{\ell_{q,\infty}} \|P\|_{\mathcal{P}(^m\ell_r^n)} \sum_{j_{m-2}=1}^{n} |z^{(m-2)}_{j_{m-2}}| \Big((j_{m-2})^{1-\frac{1}{q}} \|z\|_{\ell_{q,\infty}}\Big)^{m-3} \|z^{(m-1)}\|_{\ell_{q,\infty}} (r'+1) j_{m-2}^{\frac{2}{q'}-\frac{1}{r'}} \\
&\leqslant (r'+1) Cm^{e^{r'}} \|w\|_{\ell_{q,\infty}} \|z\|^{m-3}_{\ell_{q,\infty}} \|z^{(m-2)}\|_{\ell_{q,1}} \|z^{(m-1)}\|_{\ell_{q,\infty}} \|P\|_{\mathcal{P}(^m\ell_r^n)},
\end{aligned}$$

where in the penultimate inequality we used the bound of the identity from ℓ_1^k to $\ell_{q,\infty}^k$ that may be found for example in [DM06, Lemma 22]. On the other hand, we also have,

$$\begin{aligned}
&\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P) z_{j_1}\dots z_{j_{m-3}} z^{(m-2)}_{j_{m-2}} z^{(m-1)}_{j_{m-1}} w_{j_m}| \\
&\leqslant em\|P\|_{\mathcal{P}(^m\ell_r^n)} \sum_{\mathbf{j}\in\mathcal{J}(m-1,n)} |z_{j_1}\dots z_{j_{m-3}} z^{(m-2)}_{j_{m-2}} z^{(m-1)}_{j_{m-1}}| \cdot \Big(\frac{(m-1)^{m-1}}{\alpha(\mathbf{j})^{\alpha(\mathbf{j})}}\Big)^{1/r} \Big(\sum_{j_m=j_{m-1}}^{n} w_{j_m}^r\Big)^{1/r} \\
&\leqslant em^3 Cm^{e^{r'-1}} \|P\|_{\mathcal{P}(^m\ell_r^n)} \sum_{j_{m-1}=1}^{n} |z^{(m-1)}_{j_{m-1}}| \Big(\sum_{\mathbf{j}\in\mathcal{J}(m-3,j_{m-2})} |\mathbf{j}||z_{\mathbf{j}}|\Big) \sum_{j_{m-2}=1}^{j_{m-1}} |z^{(m-2)}_{j_{m-2}}| \Big(\sum_{j_m=j_{m-1}}^{n} w_{j_m}^r\Big)^{1/r} \\
&\leqslant Cm^{e^{r'}} \|w\|_{\ell_{q,\infty}} \|P\|_{\mathcal{P}(^m\ell_r^n)} \sum_{j_{m-1}=1}^{n} |z^{(m-1)}_{j_{m-1}}| \Big(\sum_{l=1}^{j_{m-2}} |z_l|\Big)^{m-3} \sum_{j_{m-2}=1}^{j_{m-1}} |z^{(m-2)}_{j_{m-2}}| j_{m-1}^{\frac{1}{r}-\frac{1}{q}} \\
&\leqslant Cm^{e^{r'}} \|w\|_{\ell_{q,\infty}} \|P\|_{\mathcal{P}(^m\ell_r^n)} \sum_{j_{m-1}=1}^{n} |z^{(m-1)}_{j_{m-1}}| \Big((j_{m-2})^{1-\frac{1}{q}} \|z\|_{\ell_{q,\infty}}\Big)^{m-3} j_{m-1}^{1-\frac{1}{q}} \|z^{(m-2)}\|_{\ell_{q,\infty}} j_{m-1}^{\frac{1}{r}-\frac{1}{q}} \\
&= Cm^{e^{r'}} \|w\|_{\ell_{q,\infty}} \|z\|^{m-3}_{\ell_{q,\infty}} \|z^{(m-2)}\|_{\ell_{q,\infty}} \|z^{(m-1)}\|_{\ell_{q,1}} \|P\|_{\mathcal{P}(^m\ell_r^n)}.
\end{aligned}$$

Thus, proceeding as in Lemma 8.2.4 we may construct an operator which is bounded from $\ell^d_{q,\infty}$ to $\big(\ell_{q,1}\big)'$ and also from $\ell^d_{q,1}$ to $\big(\ell_{q,\infty}\big)'$. Applying the K-interpolation method restricted to the cone of non-increasing sequences to this operator we can conclude that for any $z = z^*$,

$$\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P) z_{\mathbf{j}}| \leqslant \sqrt{(1+r')} Cm^{e^{r'}} \|z\|^{m-2}_{\ell_{q,\infty}} \|z\|^2_{\ell_{q,2}} \|P\|_{\mathcal{P}(^m\ell_r^n)} \leqslant C(1+\varepsilon)^m \|P\|_{\mathcal{P}(^m\ell_r^n)} \|z\|^m_{\ell_q^n}.$$

Therefore, by Corollary 3.3.6, we have proved our claim.

With some extra work it can proved, in a similar way, that given any $s \geqslant 1$ and $\varepsilon > 0$, there exist some m_0 and some $C > 0$ such that for every $n \in \mathbb{N}$, all $m \geqslant m_0$ and every polynomial $P \in \mathcal{P}(^m\mathbb{C}^n)$ we have

$$\sum_{\mathbf{j}\in\mathcal{J}(m,n)} |c_{\mathbf{j}}(P)z_{\mathbf{j}}| \leqslant C(1+\varepsilon)^m \|P\|_{\mathcal{P}(^m\ell_r^n)}\|z\|^m_{\ell^n_{q,s}}.$$

8.3 Multipliers

A sequence $(a_n)_{n\in\mathbb{N}}$ is a multiplier for $mon\mathcal{P}(^m\ell_r)$ if

$$(a_n)_{n\in\mathbb{N}} \cdot \ell_r \subset mon\mathcal{P}(^m\ell_r),$$

where the product $(a_n)_{n\in\mathbb{N}}\cdot\ell_r$ is just the coordinate-wise multiplication. Let $p = (p_1, p_2, \dots)$ be the sequence of the prime numbers. It is well-known that for $r \geqslant 2$, the sequence $\frac{1}{p^{\frac{m-1}{2m}}}$ is a multiplier for $mon\mathcal{P}(^m\ell_r)$ (this can be seen as an immediate consequence of [BDS19, Theorem 5.1 (3)]).

For $1 < r < 2$ in [BDS19, Theorem 5.3.] the authors prove this up to an ε, showing that for each m and every $\varepsilon > \frac{1}{r}$

$$\frac{1}{p^{\sigma_m}\big(\log(p)\big)^{\varepsilon}} \cdot \ell_r \subset mon\mathcal{P}(^m\ell_r), \tag{8.14}$$

where $\sigma_m = \frac{m-1}{m}\left(1-\frac{1}{r}\right)$. As a consequence of our results, we can improve this, showing that, for $1 < r \leqslant 2$, even the sequence $(\frac{1}{n^{\sigma_m}})_{n\in\mathbb{N}}$ is a multiplier for $mon\mathcal{P}(^m\ell_r)$.

Theorem 8.3.1. *For $1 < r < 2$ and $m \geqslant 3$ put $\sigma_m = \frac{m-1}{m}\left(1-\frac{1}{r}\right)$. Then,*

$$\Big(\frac{1}{n^{\sigma_m}}\Big)_n \cdot \ell_r \subset mon\mathcal{P}(^m\ell_r),$$

and σ_m is best possible.

Proof. As a consequence of Theorem 8.0.1 we know that $\ell_{q,r} \subset mon\mathcal{P}(^m\ell_r)$, thus to prove the result it is sufficient to see that if $z \in \ell_r$ then, $\big(\frac{1}{n^{\sigma_m}}\big)_n \cdot z \in \ell_{q,r}$. Suppose that $z \in \ell_r$ is an arbitrary element (not necessarily equal to z^*). Since $r > q$ we know that the semi-norm $\|\cdot\|_{\ell_{q,r}}$ is equivalent to the following maximal norm (as stated in (1.2))

$$\|w\|_{\ell_{(q,r)}} = \left(\sum_{n=1}^{\infty} n^{\frac{r}{q}-1}\left(\frac{1}{n}\sum_{k=1}^{n} w_k^*\right)^r\right)^{1/r}.$$

Then, if $w = \big(\frac{z_n}{n^{\sigma_m}}\big)_n$, by the *Hardy-Littlewood rearrangement inequality* (Lemma 6.3.5) it is easy to see that

$$\sum_{k=1}^{n} w_k^* \leqslant \sum_{k=1}^{n} z_k^* \frac{1}{k^{\sigma_m}}$$

for every $n \in \mathbb{N}$. Then

$$\Big\| \Big(\frac{z_n}{n^{\sigma_m}} \Big)_n \Big\|_{\ell_{q,r}} \sim \Big\| \Big(\frac{z_n}{n^{\sigma_m}} \Big)_n \Big\|_{\ell_{(q,r)}} \leqslant \left(\sum_{n=1}^{\infty} n^{\frac{r}{q}-1} \left(\frac{1}{n} \sum_{k=1}^{n} z_k^* \frac{1}{k^{\sigma_m}} \right)^r \right)^{1/r}$$

$$= \left\| \left(\frac{z_n^*}{n^{\sigma_m}} \right)_n \right\|_{\ell_{(q,r)}} \sim \left\| \left(\frac{z_n^*}{n^{\sigma_m}} \right)_n \right\|_{\ell_{q,r}} = \left(\sum_{n=1}^{\infty} \left((\frac{z_n^*}{n^{\sigma_m}})^* n^{\frac{1}{q}-\frac{1}{r}} \right)^r \right)^{1/r} = \|z\|_{\ell_r} < \infty,$$

where, in the last equality, we have used the fact that $\sigma_m = \frac{1}{q} - \frac{1}{r}$.
To see that the exponent is optimal take, as always, $q = (mr')'$. Now, if $(z_n)_n = \left(\frac{1}{n^{1/r} \log(n+1)^{2/r}} \right)_n \in \ell_r$ for every $\varepsilon > 0$ it is easy to check that the sequence $\left(\frac{z_n}{n^{\sigma_m - \varepsilon}} \right)_n \notin \ell_{q,\infty} \supset mon\mathcal{P}({}^m\ell_r)$. $\square$

For $m = 2$ we cannot show that the sequence $\left(\frac{1}{n^{\sigma_2}} \right)_n$ is a multiplier for $mon\mathcal{P}({}^2\ell_r)$ but using the fact that $\ell_q \subset mon\mathcal{P}({}^2\ell_r)$, Theorem 8.0.1, it is easy to see that we have the inclusion

$$\frac{1}{p^{\sigma_2} \big(\log(p) \big)^{\varepsilon}} \cdot \ell_r \subset mon\mathcal{P} \left({}^2\ell_r \right),$$

for *every* $\varepsilon > 0$ extending [BDS19, Theorem 5.3.] (see also (8.14)). We leave the details to the reader.

Appendix A

Monomial series expansion

We dedicate this appendix to the proof of Proposition 3.1.2:

Proposition A.0.1 (Proposition 3.1.2)**.** *Given* $1 < p, q \leqslant \infty$, *for* $X = \ell_{p,q}$ *we have*

$$mon\mathcal{F}(\mathcal{R}) = \Big\{ z \in \mathcal{R} : f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z^\alpha \;\; \textit{for every } f \in \mathcal{F}(\mathcal{R}) \Big\},$$

for $\mathcal{F}(\mathcal{R})$ *being* $H_b(X), H_\infty(B_X)$ *or* $\mathcal{P}(^m X)$ *for any* $m \in \mathbb{N}$.

Recall that, in order to make sense, the convergence for the monomial expansion of a given holomorphic function needs to be unconditional. As the unconditional and the absolute convergence are equivalent concepts in $\mathbb{C}$, for every family of holomorphic function on a Reinhardt domain $\mathcal{F}(\mathcal{R})$ it holds

$$\Big\{ z \in \mathcal{R} : f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z^\alpha \;\; \text{for every } f \in \mathcal{F}(\mathcal{R}) \Big\} \subset mon\mathcal{F}(\mathcal{R}). \tag{A.1}$$

Now we will give some results needed to prove the reverse inclusion.

Given a Banach sequence space X we may consider its Köthe dual space

$$X^\times := \left\{ y \in \mathbb{C}^{\mathbb{N}} : \sum_{k \geqslant 1} |y_k x_k| < \infty \text{ for all } x \in X \right\},$$

which endowed with the norm

$$\|y\|_{X^\times} := \sup_{x \in B_X} \sum_{k \geqslant 1} |y_k x_k|,$$

is a symmetric Banach sequence space. It is worth mentioning that for $1 \leqslant p < 1$ it holds $\ell_p^\times = \ell_{p'}$ and also $\ell_\infty^\times = \ell_1$. This shows in particular that the Köthe dual and the classical dual given by the continuous functionals do not always coincide. Observe that for every Banach sequence space X it holds $X^\times \subset X'$ isometrically.

As for every Banach sequence space X its Köthe dual is again a Banach sequence space we may consider $(X^\times)^\times$. We will simply denote it by $X^{\times\times}$. It is easy to see $X \subset X^{\times\times}$.

The following is a classic result in the theory of Banach sequence spaces, a proof of this fact can be found in [Maz10, Teorema 1.3.11].

Remark A.0.2. For every Banach sequence space X the following are equivalent:

a) $[e_n : n \in \mathbb{N}]$ is dense in X.

b) X is separable.

c) $X' = X^\times$.

A corollary of the previous remark states that for a separable Banach sequence space X it holds

$$(X')^\times = X^{\times\times}. \tag{A.2}$$

Remark A.0.3. Given a separable Banach sequence space X, the set of monomial convergence for the family of linear functionals $X' = \mathcal{L}(X) = \mathcal{P}(^1X)$ is the Köethe dual of X', i.e.,

$$monX' = mon\mathcal{P}(^1X) = (X')^\times = X^{\times\times}.$$

Proof. First, since X is separable, by Remark A.0.2 we have $X' = X^\times$. This allows us to think about X' as Banach sequence space itself. Given $\phi \in X'$ we may associate it to a sequence $(\phi(e_k))_{k\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}}$.

On the other hand, for every $\phi \in X'$, as it is a 1-homogeneous polynomial, we have that $a_\alpha(\phi) = 0$ for $|[\alpha]| \neq 1$. It is easy to see that $a_\alpha(\phi) = \phi(e_k)$ if $\alpha = e_k$ for every $k \in \mathbb{N}$. For $z \in \mathbb{C}^{\mathbb{N}}$ and $\phi \in X'$ it holds

$$\sum_{k\geqslant 1} |\phi(e_k)z_k| = \sum_{\alpha\in\Lambda(1)} |a_\alpha(\phi)z^\alpha|. \tag{A.3}$$

Then, if $z \in (X')^\times$ the left hand side in equation (A.3) sums (is finite) and therefore $z \in monX'$. Reversely, $z \in monX'$ implies that the right hand side is finite for every $\phi \in X'$ and then $z \in (X')^\times$. This shows that $monX' = (X')^\times = X^{\times\times}$ using (A.2). □

We will see below that, for reflexive and separable Banach sequence spaces our purpose will be achievable thanks to the previous remark. This is the case of $\ell_{p,q}$ with $1 < p < \infty$ and $1 \leqslant q < \infty$ by Theorem 1.1.4. For $\ell_{p,\infty}$ with $1 < p < \infty$ we will need the following lemma.

Lemma A.0.4. *For $1 < p < \infty$ we have $\ell_{p,\infty}^\times = \ell_{p',1}$.*

Proof. We begin by proving $\ell_{p,\infty}^\times \subset \ell_{p',1}$. Take $z \in \ell_{p,\infty}^\times$, then

$$\begin{aligned}\|z\|_{\ell_{p',1}} &= \|(k^{\frac{1}{p'}-1}z_k*)_{k\geqslant 1}\|_{\ell_1}\\ &= \sum_{k\geqslant 1} |k^{-1/p}z_k^*| < \infty,\end{aligned}$$

since $(k^{-1/p})_{k\geqslant 1} \in \ell_{p,\infty}$ and $z^* \in \ell_{p,\infty}^\times$ as it is a symmetric Banach sequence space. We have proved that $z \in \ell_{p',1}$.

On the other hand, let $z \in \ell_{p',1}$ and take any $x \in \ell_{q,\infty}$. Using the *Hardy-Littlewood rearrangement inequality* (Lemma 6.3.5) it follows (using Hölder inequality),

$$\begin{aligned} \sum_{k\geqslant 1} |z_k x_k| &\leqslant \sum_{k\geqslant 1} z_k^* x_k^* \\ &= \sum_{k\geqslant 1} k^{-1/p} z_k^* k^{1/p} x_k^* \\ &\leqslant \|z\|_{\ell_{p',1}} \|x\|_{\ell_{p,\infty}} < \infty, \end{aligned}$$

then $z \in \ell_{p,\infty}^\times$. □

We will summarize some of the previous results in the following lemma.

Lemma A.0.5. *As long as $X = \ell_{p,q}$ with $1 < p < \infty$ and $1 < q \leqslant \infty$ it follows*

$$monX' = mon\mathcal{P}(^1X) = X^{\times\times} = X.$$

Proof. Let $X = \ell_{p,q}$ with $1 < p, q < \infty$, by Theorem 1.1.4 X is reflexive. Since it is also separable, thanks to Remark A.0.2 and Remark A.0.3, it holds

$$mon\mathcal{P}(^1X) \subset (X')^\times = X^{\times\times} = (X')' = X.$$

Finally let us take $X = \ell_{p,\infty}$ with $1 < p < \infty$, which is also separable. Using Remark A.0.3 and the inequality in (A.2) we have

$$mon\mathcal{P}(^1X) = X^{\times\times} \subset (X^\times)'.$$

By Lemma A.0.4 and Theorem 1.1.4 we have

$$(X^\times)' = (\ell_{p',1})' = \ell_{p,\infty} = X,$$

then it follows $X^{\times\times} \subset (X^\times)' \subset X$. Since for every Banach sequence space $X \subset X^{\times\times}$ we have what we wanted. □

For every separable Banach sequence space X it holds that $X' \subset H_b(X) \subset H_\infty(B_X)$ and (3.3) gives that

$$monH_\infty(B_X) \subset monH_b(X) \subset monX' = X^{\times\times}.$$

In particular, for those X such that $X = X^{\times\times}$, we have

$$mon\mathcal{F} = \left\{ z \in X : \sum_{\alpha\in\mathbb{N}_0^{(\mathbb{N})}} |a_\alpha(f) z^\alpha| < \infty \text{ for every } f \in \mathcal{F} \right\}, \tag{A.4}$$

for $\mathcal{F}$ being $H_\infty(B_X)$ or $H_b(X)$. It should be noted that the importance in the previous equality is the fact that all the elements in $mon\mathcal{F}$ lie in X.

The same conclusion suits for the family of homogeneous polynomial, but first we need the following proposition [DGMSP19, Remark 10.7]. We give the proof here for completeness.

Proposition A.0.6. *Given X a Banach sequence space for every $m \in \mathbb{N}$ it holds*

$$mon\mathcal{P}(^{m+1}X) \subset mon\mathcal{P}(^{m}X).$$

Proof. Fix $u \in mon\mathcal{P}(^{m+1}X)$ not null and $P \in \mathcal{P}(^{m}X)$. Choose $i \in \mathbb{N}$ such that $u_i \neq 0$ and define $Q \in \mathcal{P}(^{m+1}X)$ by $Q(z) = z_i P(z)$. It is easy to see that

$$a_\alpha(Q) = \begin{cases} 0 & \text{if } \alpha_i = 0 \\ a_{\tilde{\alpha}}(P) & \text{if } \alpha_i > 0, \end{cases}$$

where $\tilde{\alpha}_j = \alpha_j + \delta_{i,j}$, then

$$\begin{aligned} \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |a_\alpha(P) u^\alpha| &= \frac{1}{u_i} \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |a_\alpha(P) u^\alpha u_i| \\ &= \frac{1}{u_i} \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |a_{\tilde{\alpha}}(P) u^{\tilde{\alpha}}| \\ &= \frac{1}{u_i} \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |a_\alpha(Q) u^\alpha| < \infty. \end{aligned}$$

□

Now, given $m \in \mathbb{N}$, and using that $mon\mathcal{P}(^{m}X) \subset \mathcal{P}(^{1}X) = (X')^\times$ (which simply follows from Proposition A.0.6) and proceeding as before we have

$$mon\mathcal{P}(^{m}X) = \left\{ z \in X : \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |a_\alpha(f) z^\alpha| < \infty \text{ for all } f \in \mathcal{P}(^{m}X) \right\}, \tag{A.5}$$

for every separable Banach sequence space X such that $X^{\times\times} = X$.

The following proposition essentially states that, given a separable Banach sequence space X, any element in the set of monomial convergence for the family $H_\infty(B_X)$ must be inside the ball of X, its proof is given in [DGMSP19, Proposition 20.3].

Proposition A.0.7. *Given a Banach sequence space X, it holds $X \cap monH_\infty(B_X) \subset B_X$.*

As we said before, if we use Proposition A.0.7 and Remark A.0.2, for any separable Banach sequence space such that $X^{\times\times} = X$ we have

$$monH_\infty(B_X) = \left\{ z \in B_X : \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |a_\alpha(f) z^\alpha| < \infty \text{ for all } f \in H_\infty(B_X) \right\}. \tag{A.6}$$

Remark A.0.8. Let X be a separable Banach sequence space such that $X^{\times\times} = X$ then

$$mon\mathcal{F}(\mathcal{R}) = \left\{ z \in \mathcal{R} : \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} |a_\alpha(f) z^\alpha| < \infty \text{ for every } f \in \mathcal{F} \right\},$$

for $\mathcal{F}(\mathcal{R})$ being $H_\infty(B_X), H_b(X)$ or $\mathcal{P}(^{m}X)$ for every $m \in \mathbb{N}$ and, of course, $\mathcal{R}$ being B_X or X respectively.

Note that Remark A.0.8 holds using (A.4) for $H_b(X)$, (A.5) for $\mathcal{P}(^m X)$ and (A.6) for $H_\infty(B_X)$.

In [BDF+17, Equation (15)] the authors state that

$$mon H_\infty(B_{c_0}) = mon H_\infty(B_{\ell_\infty}) \text{ and } mon\mathcal{P}(^m c_0) = mon\mathcal{P}(^m \ell_\infty). \tag{A.7}$$

There they cite [DMP09, Remark 6.4] where it is shown the first assertion in (A.7). The argument used there comes from a result in [DG89]. In that article Davie and Gamelin show that every $f \in H_\infty(B_X)$ can be extended to some $\widehat{f} \in H_\infty(B_{X''})$ without changing its norm. For $f \in H_b(X)$ it is standard that this can also be done and again implies

$$mon H_b(c_0) = mon H_b(\ell_\infty). \tag{A.8}$$

Now thanks to the equations in (A.7) and by Theorem 3.2.2 and Theorem 3.2.3 it follows

$$\begin{aligned} mon\mathcal{P}(^m c_0) &= mon\mathcal{P}(^m \ell_\infty) \subset c_0, \\ mon H_\infty(B_{c_0}) &= mon H_\infty(B_{\ell_\infty}) \subset \overline{B_{c_0}}. \end{aligned} \tag{A.9}$$

Finally by equation (A.8) and Theorem 6.3.2 we have

$$mon H_b(c_0) = mon H_b(\ell_\infty) \subset c_0. \tag{A.10}$$

The following proposition is a slightly more general version of Proposition 3.1.2, which we aim to prove in this appendix.

Proposition A.0.9. *Let X be ℓ_∞ or any separable Banach sequence space X such that $X^{\times\times} = X$, it holds*

$$mon\mathcal{F}(\mathcal{R}) = \left\{ z \in \mathcal{R} : f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z^\alpha \text{ for every } f \in \mathcal{F}(\mathcal{R}) \right\},$$

for $\mathcal{F}(\mathcal{R})$ being $H_\infty(B_X), H_b(X)$ or $\mathcal{P}(^m X)$ for every $m \in \mathbb{N}$ and, of course, $\mathcal{R}$ being B_X or X respectively.

Proof. By (A.1) it is only left to prove

$$mon\mathcal{F}(\mathcal{R}) \subset \left\{ z \in \mathcal{R} : f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z^\alpha \text{ for every } f \in \mathcal{F} \right\}.$$

Fix $z \in mon\mathcal{F}(\mathcal{R})$. In the case of X being ℓ_∞ by equations (A.9) and (A.10) we have $z \in c_0$ or $z \in B_{c_0}$ depending on $\mathcal{F}(\mathcal{R})$. When X is a separable Banach sequence space $X^{\times\times} = X$ using Remark A.0.8 we have $z \in \mathcal{R}$.

Every $f \in \mathcal{F}(\mathcal{R})$ is continuous in $\mathcal{R}$ as it is holomorphic there. Since $[e_n : n \in \mathbb{N}]$ is dense in c_0 or X we have $\pi_n(z) \to z$, when $n \to \infty$, and then $f(\pi_n(z)) \to f(z)$. On the other hand, as $\pi_n(z) \in \mathcal{R}_n$ it holds

$$f(\pi_n(z)) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha(f) z^\alpha,$$

and $z \in mon\mathcal{F}(\mathcal{R})$ so the monomial expansion converges absolutely in z, all this together implies

$$\lim_{n\to\infty} f(\pi_n(z)) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z^\alpha.$$

By the uniqueness of the limit we have $f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z^\alpha$. □

Now we are able to achieve the goal of this appendix. The proof follows directly from what we saw.

Proof of Proposition 3.1.2. Thanks to Lemma A.0.5 it holds that $X^{\times\times} = X$ for $X = \ell_{p,q}$ with $1 < p < \infty$ and $1 < q \leqslant \infty$. As X is separable using Proposition A.0.9 it holds

$$mon\mathcal{F}(\mathcal{R}) = \left\{ z \in \mathcal{R} : f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} a_\alpha(f) z^\alpha \text{ for every } f \in \mathcal{F}(\mathcal{R}) \right\}.$$

For $X = \ell_\infty$ this is explicitly proved in Proposition A.0.9. □

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