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Razón de volumen entre cuerpos convexos

Tesis presentada para optar al título de Doctor de la Universidad de Buenos Aires en el área Ciencias Matemáticas

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Resumen

Esta tesis tiene como objeto contribuir al estudio de algunos problemas de análisis geométrico asintótico relativos a aproximaciones volumétricas de un cuerpo convexo mediante imágenes afines de otro.

Dado un cuerpo convexo $K \subset \mathbb{R}^n$ con baricentro en el origen, mostramos que existe un símplice $S \subset K$ que tiene también *baricentro en el origen* tal que $\left(\frac{|S|}{|K|}\right)^{1/n} \geqslant \frac{c}{\sqrt{n}}$, donde $c > 0$ es una constante absoluta y $|\cdot|$ denota la medida de Lebesgue. Conseguimos esto usando técnicas de geometría estocástica. Más precisamente, si K está en posición isotrópica, presentamos un método para encontrar símplices centrados verificando la cota antes mencionada que funciona con probabilidad extremadamente alta.

Por dualidad, dado un cuerpo convexo $K \subset \mathbb{R}^n$ mostramos que existe un símplice S que contiene a K *con el mismo baricentro* tal que $\left(\frac{|S|}{|K|}\right)^{1/n} \leqslant d\sqrt{n}$, para alguna constante absoluta $d > 0$. Salvo por la constante la estimación no puede ser mejorada.

Defimos la *máxima razón de volumen* de un cuerpo convexo $K \subset \mathbb{R}^n$ como $\mathrm{lvr}(K) := \sup_{L \subset \mathbb{R}^n} \mathrm{vr}(K, L)$, donde el supremo se toma sobre todos los cuerpos convexos L. Probamos la siguiente cota que resulta ajustada en general: $c\sqrt{n} \leqslant \mathrm{lvr}(K)$, para *todo* cuerpo K (donde $c > 0$ es una constante absoluta). Este resultado mejora la cota anteriormente conocida que es del orden de $\sqrt{\frac{n}{\log\log(n)}}$.

Estudiamos el comportamiento asintótico exacto para algunas clases naturales de cuerpos convexos. En particular, si K es la bola unitaria de una norma unitariamente invariante en $\mathbb{R}^{d\times d}$ (e.g., la bola unidad de la clase p-Schatten para $1 \leqslant p \leqslant \infty$), la bola unidad de una norma tensorial en el producto de espacios ℓ_p o K un cuerpo incondicional, probamos que $\mathrm{lvr}(K)$ se comporta como la raíz cuadrada de la dimensión del espacio ambiente

También analizamos el problema de estimar la razón de volumen entre proyecciones de dos cuerpos convexos en $\mathbb{R}^n$ en subespacios de dimensión proporcional a n.

Palabras clave: Razón de volumen, simplices, cuerpos convexos, politopos aleatorios.

Abstract

This thesis aims to contribute to the study of some problems of asymptotic geometrical analysis concerning volumetric approximations of a convex body by an affine image of another one. For a convex body $K \subset \mathbb{R}^n$ with barycenter at the origin, we show that there is a simplex $S \subset K$ having also *barycenter at the origin* such that $\left(\frac{|S|}{|K|}\right)^{1/n} \geqslant \frac{c}{\sqrt{n}}$, where $c > 0$ is an absolute constant and $|\cdot|$ stands for the Lebesgue measure. This is achieved using stochastic geometric techniques. More precisely, if K is in isotropic position, we present a method to find centered simplices verifying the above bound that works with extremely high probability. By duality, given a convex body $K \subset \mathbb{R}^n$ we show that there is a simplex S enclosing K *with the same barycenter* such that $\left(\frac{|S|}{|K|}\right)^{1/n} \leqslant d\sqrt{n}$, for some absolute constant $d > 0$. Up to the constant, the estimate cannot be lessened.

We define the *largest volume ratio* of given convex body $K \subset \mathbb{R}^n$ as $\mathrm{lvr}(K) := \sup_{L \subset \mathbb{R}^n} \mathrm{vr}(K, L)$, where the sup runs over all the convex bodies L. We prove the following sharp lower bound: $c\sqrt{n} \leqslant \mathrm{lvr}(K)$, for *every* body K (where $c > 0$ is an absolute constant). This result improves the former best known lower bound, of order $\sqrt{\frac{n}{\log\log(n)}}$.

We study the exact asymptotic behaviour of the largest volume ratio for some natural classes of convex bodies. In particular, if K is the unit ball of an unitary invariant norm in $\mathbb{R}^{d\times d}$ (e.g., the unit ball of the p-Schatten class S_p^d for any $1 \leqslant p \leqslant \infty$), the unit ball of a tensor norm on the product of ℓ_p spaces or K is unconditional, we show that $\mathrm{lvr}(K)$ behaves as the square root of the dimension of the ambient space.

We also analyse the problem of estimating the volume ratio between projections of two bodies in $\mathbb{R}^n$ onto subspaces of dimension proportional to n.

Keywords: Volume ratio, simplices, convex bodies, random polytopes.

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Contents

Introduction

Given a real vector space of finite dimension endowed with a norm, its unit ball is a convex, compact set with non-empty interior (what we call a *convex body*). On the other hand, through Minkowski's functional, any centrally symmetric convex body in $\mathbb{R}^n$ is the unit ball for some norm. So, there is a close connection between the geometry and the metric structure of Banach spaces.

Classically, geometry was studied in low dimensions, usually two or three. The study of geometrical properties of finite dimensional Banach spaces of high dimension had a large development during the last decades, while trying to understand infinite dimensional spaces. Later, the geometry of spaces of high dimension aroused interest by itself. In this setting one studies families of objects in different spaces focused on the asymptotic behaviour of certain quantities. Usually the dependence relies on the the dimension of the ambient space.

For many applications in asymptotic geometric analysis, convex geometry or even optimization it is useful to approximate a given convex body by another one. For example, the classical Rogers-Shephard inequality [AAGM15, Theorem 1.5.2] states that, for a convex body $K \subset \mathbb{R}^n$, the volume of the difference body $K - K$ is "comparable" with the volume of K. Precisely, $|K - K|^{\frac{1}{n}} \leqslant 4|K|^{\frac{1}{n}}$ where $|\cdot|$ stands for the n-dimensional Lebesgue measure. Rogers and Shephard also showed, with the additional assumption that K has barycenter at the origin, that the intersection body $K \cap (-K)$ has "large" volume. Namely, $|K \cap (-K)|^{\frac{1}{n}} \geqslant \frac{1}{2}|K|^{\frac{1}{n}}$. These inequalities imply that any given body is enclosed by (or contains) a symmetric body whose volume is "small" ("large") enough. In many cases this allows us to take advantage of the symmetry of the difference body (or the intersection body) to conclude something about K.

Another interesting example of Milman and Pajor [MP89, Section 3] shows that

$$L_K \leqslant c \inf \left\{ \left(\frac{|W|}{|K|} \right)^{\frac{1}{n}} : W \text{ is unconditional and contains } K \right\}, \qquad (1)$$

where L_K stands for the isotropic constant of $K \subset \mathbb{R}^n$ (see [BGVV14, Section 2.3.1]) and $c > 0$ is an absolute constant. Therefore, having a good volumetric approximation of K by an unconditional convex body provides structural geometric information of K.

Perhaps the most notable application of these kind of approximations can be viewed when studying John/Löwner ellipsoid (maximum/minimum volume ellipsoid respectively). John proved that, if the Euclidean ball is the maximal volume ellipsoid inside K, we can decompose the identity as a linear combination of rank-one operators defined by contact points [AAGM15, Theorem 2.1.10]. This result was complemented by Ball [Bal92] who showed that this property characterizes Jonh's ellipsoid. The distribution of the contact points between a convex body and its maximal volume ellipsoid was also used by Dvoretzky and Rogers [DR50] to prove that every infinite dimensional Banach space has a series that converges unconditionally but not absolutely. It also plays a key role in the study of distances between bodies, see [TJ89] for a complete treatment on this. We also refer to [Mat02, Gru07, GPT01, Las92, Las98, Pel83] for many nice results/applications which involve these extremal ellipsoids.

A natural quantity that relates a given body K with its ellipsoid of maximal volume is given by the "standard" *volume ratio*, that was introduced by Szarek and Tomczak Jaegermanb in [STJ80]

$$\mathrm{vr}(K) = \inf \left\{ \left(\frac{|K|}{|\mathcal{E}|} \right)^{\frac{1}{n}} : \mathcal{E} \text{ is an ellipsoid contained in } K \right\}. \tag{2}$$

Using the Brascamp-Lieb inequality, Ball showed that $\mathrm{vr}(K)$ is maximal when K is a simplex . The extreme case, among all the centrally symmetric convex bodies, is given by the cube (see [AAGM15, Theorem 2.4.8]).

A generalization of the "standard" volume ratio was presented by Giannopoulos and Hartzoulaki [GH02] and also developed by Gordon, Litvak, Meyer and Pajor [GLMP04]: given two convex bodies K and L in $\mathbb{R}^n$ the *volume ratio* of the pair (K, L) is defined as

$$\mathrm{vr}(K, L) := \inf \left\{ \left(\frac{|K|}{|T(L)|} \right)^{\frac{1}{n}} : T(L) \text{ is contained in } K \right\}, \tag{3}$$

where the infimum (actually a minimum) is taken over all affine transformations T. In other words, $\mathrm{vr}(K, L)$ measures how well can K be approximated by an affine image of L. Note that the classic value $\mathrm{vr}(K)$ is just $\mathrm{vr}(K, B_2^n)$ where B_2^n is the Euclidean unit ball in $\mathbb{R}^n$. It's easy to see that the volume ratio is invariant under affine transformations, which means that it depends only on the affine classes of K and L. This invariant can already be found in the work of McBeath [Mac51a]. The cubical volume ratio, $\mathrm{vr}(B_\infty^n, K)$, was studied by Ball [Bal89] who proved that for every convex body $K \subset \mathbb{R}^n$,

$$\mathrm{vr}(B_\infty^n, K)\,\mathrm{vr}(K, B_2^n) \sim \mathrm{vr}(B_\infty^n, B_2^n).$$

It also appears in the work of Babenko [Bab88] (under the name of supporting volume) and also was studied by Pelczynski and Szarek in [PS91]. Actually, a bound for the cubical volume ratio can be already found in [DR50].

We treat the problem of bounding the simplex volume ratio for a convex body K. Namely,

$$S^{out}(K) := \mathrm{vr}(S, K)$$

where S is a simplex (the convex hull of $n+1$ affinely independent points on $\mathbb{R}^n$). Given $K \subset \mathbb{R}^n$ we look for simplices containing it with "small" volume. All this generalizes, for higher dimension, a classical geometrical problem: given a convex set $K \subset \mathbb{R}^2$ finding the minimal area of a triangle containing it. In [Gro18], Gross proved that for every convex body $K \subset \mathbb{R}^2$ there is a triangle of at most twice the area containing it. For greater dimensions the problem of finding the exact value of the simplex ratio remains open.

Macbeath in [Mac51a] showed how to construct a simplex that contains a convex body $K \subset \mathbb{R}^n$ such that $|S| \leqslant n^n |K|$, obtainting that $S^{out}(K) \leqslant n$. Chakerian [Cha73, Corollary 5] improved this bound showing that

$$S^{out}(K) \leqslant n^{\frac{n-1}{n}} \approx n. \tag{4}$$

The best known estimate so far can be obtained applying a general bound for volume ratios due to Giannopoulos and Hartzoulaki [GH02],

$$S^{out}(K) \preceq \sqrt{n} \log(n). \tag{5}$$

In this work we show the asymptotically sharp bound

$$S^{out}(K) \preceq \sqrt{n}.$$

Actually, we prove something stronger (Theorem 2.3.4): given $K \subset \mathbb{R}^n$ there is a simplex enclosing it with the *same barycenter* such that

$$\left(\frac{|S|}{|K|}\right)^{\frac{1}{n}} \preceq \sqrt{n}.$$

We work with a dual version of this problem and prove that, given a body $K \subset \mathbb{R}^n$, there is a simplex contained in it with the *same barycenter* such that

$$\left(\frac{|K|}{|S|}\right)^{\frac{1}{n}} \preceq \sqrt{n}.$$

The techniques that we use allow us to obtain a result of probabilistic nature, Theorem 2.4.6, that can be seen as a random algorithm to find such simplices.

In order to face the problem for other classes of convex bodies we define the *largest volume ratio* of a convex body K as

$$\operatorname{lvr}(K) := \sup_{L \subset \mathbb{R}^n} \operatorname{vr}(K, L),$$

where the supremum is taken over all convex bodies $L \subset \mathbb{R}^n$. The bound (5) can be written as

$$\operatorname{lvr}(K) \preceq \sqrt{n} \log(n), \tag{6}$$

for every convex body $K \subset \mathbb{R}^n$. In many cases the largest volume ratio of a body can be bounded by the square root of the dimension of the ambient space. In fact we conjecture that the logarithmic factor in (6) can be removed. We show this bound for many natural classes of convex bodies.

We study the case where $K \subset \mathbb{R}^{d \times d}$ is the unit ball of the p-Schatten norm. This norms are generalizations of the classical Hilbert-Schmidt operator norm. We refer to [KMP98, GP07, BCE13, RV16, KPT18] where many properties of them can be found. These examples come from unitary invariant norms. In Theorem 3.3.6 we prove that if $K \subset \mathbb{R}^{d \times d}$ is the unit ball of a unitary invariant norm, then

$$\operatorname{lvr}(K) \preceq d.$$

Another natural class of convex body that we treat are the unit balls of tensor norms of ℓ_p^n spaces. They have been widely studied since they can be understood as spaces of multilinear forms or homogeneous polynomials (see for example [DF92, Din99, Flo97]). We study the case of the injective and projective norm and their symmetric analogous. Namely, we prove that if E is any of the spaces $\bigotimes_{\varepsilon}^{m} \ell_p^n, \bigotimes_{\varepsilon_s}^{m,s} \ell_p^n, \bigotimes_{\pi}^{m} \ell_p^n$ or $\bigotimes_{\pi_s}^{m,s} \ell_p^n$ and $N = dim(E)$ then

$$\operatorname{lvr}(B_E) \preceq \sqrt{N}.$$

Additionally, we show that if $K \subset \mathbb{R}^n$ is unconditional, then

$$\operatorname{lvr}(K) \preceq \sqrt{n}.$$

We also treat the problem of finding a lower bound for the volume ratio. Khrabrov [Khr01], using a construction due to Gluskin [Glu81], proved that for every convex body $K \subset \mathbb{R}^n$

$$\operatorname{lvr}(K) \succeq \sqrt{\frac{n}{\log\log(n)}}. \tag{7}$$

To take out the double logarithm in (7) we refine Khrabrov's techniques. We prove in Theorem 4.2.9 that for every convex body $K \subset \mathbb{R}^n$

$$\operatorname{lvr}(K) \succeq \sqrt{n}.$$

If we combine this bound with the upper bounds that we previously mentioned we see that this is the best possible asymptotic general bound.

In [Rud04] Rudelson studied the diameter of the Banach-Mazur compactum for distances related with projections or sections of convex bodies. Based in Rudelson's approach we analyse the problem of estimating the volume ratio between projections of two bodies in $\mathbb{R}^n$ onto subspaces of dimension proportional to n. We prove in Theorem 5.1.1 that for every convex body $K \subset \mathbb{R}^n$ and $k \sim n$ there is a convex body Z such that

$$\mathrm{vr}(QK, QZ) \geqslant d\sqrt{\frac{k}{\log\log k}},$$

for every orthogonal projection $Q : \mathbb{R}^n \to \mathbb{R}^n$ of rank k.

By polarity, we get a dual version of the result: for every convex body $K \subset \mathbb{R}^n$ and $k \sim n$ there is a convex body Z such that

$$\mathrm{vr}(Z \cap E, K \cap E) \geqslant d\sqrt{\frac{k}{\log\log k}},$$

for every subspace $E \subset \mathbb{R}^n$ of dimension k.

The work is organized in five chapters.

In Chapter 1 we present the notation and some basics definitions in convex geometry. We also recall some inequalities that involve the volume of convex bodies that are going to be usefull. Lastly, we introduce some classical positions (affine images) of convex bodies such as John's/Löwner's, the ℓ-position or the isotropic position. We also present the main properties of these positions that are of our interest.

In Chapter 2 we face the problem of bounding the outer simplex ratio. For that we set a dual version of it: for a convex body K, we look for "big" simplices contained in it. In order to properly establish the correspondence between this two versions we need to apply the Blaschke-Santaló inequality [AAGM15, Theorem 1.5.10], that relates the volume of a convex body with the volume of its polar. For that we need that the simplices verify an aditional condition: that they share the same barycenter with the convex body involved.

For proving that $\mathrm{vr}(K, S) \preceq \sqrt{n}$ and $\mathrm{vr}(S, K) \preceq \sqrt{n}$, for $K, S \subset \mathbb{R}^n$ an arbitrary body and a simplex respectively, we use a probabilistic method. The idea is to see that if one randomly picks elements from a specified set, with positive probability the result belongs to the prescribed class. This is what allows us to obtain Theroem 2.4.6. We also present a non-probabilistic version of the same result based on a constructions of Dvoretzky and Rogers [DR50], that has the disadvantage of requiring an explicit computation of some contact points. Based on a result of Pivovarov [Piv10] we show a probabilistic version of this construction.

Chapter 3 is dedicated to bounding the largest volume ratio for some natural classes of convex bodies. We first prove some elementary properties of this invariant and show some examples for which sharp bounds can be obtained. We review Giannopoulos and Hartzoulaki's proof of the best known general bound [GH02], which is based on a combination of a very particular position introduced by Rudelson [Rud00] together with Chevet's inequality [AAGM15, Theorem 9.4.1]. We take advantage of specific geometrical properties of some classes of convex bodies and use the mentioned techniques in order to see that, for these classes, the largest volume ratio can be bounded by the square root of the dimension of the ambient space.

In Chapter 4 we prove a lower bound for $\mathrm{lvr}(K)$, improving the best known bound so far due to Krabrov [Khr01]. We present the definition of the random polytopes introduced by Gluskin [Glu81] (and also used by Khrabrov) to prove that $\mathrm{lvr}(K) \geq \sqrt{\frac{n}{\log\log(n)}}$. We make some subtle but important changes in Khrabrov's arguments that allows us to prove that $\mathrm{lvr}(K) \geq \sqrt{n}$ in case that $K \subset \mathbb{R}^n$ has some special geometrical properties. To extend the bound for every convex body we exploit two significant results. The first one deals with concentration of mass on isotropic convex bodies and is due to Paouris [Pao06], while the second one is Klartag's solution to the isomorphic slicing problem [Kla06].

In Chapter 5 we study the volume ratio between projections of two convex bodies. Given $K \subset \mathbb{R}^n$ and k proportional to n, we prove the existence of a body Z such that, for any orthogonal projection Q of rank k, the volume ratio between QK and QZ is "large". Some technicalities in order to deal with every projection are overcome using an ε-net argument and a Gaussian version of the random polytopes.

Chapter 1

Preliminaries

In this chapter we present the background material concerning the theory of convex bodies and asymptotic geometrical analysis that we are going to use through the entire work. We introduce some basic definitions from classical convexity and set some notation. We also state some important volume inequalities that will play a key role in order to obtain our main results. In Section 1.3 we discuss particular positions of convex bodies that showed to be especially useful. For a complete discussion on the subjects treated in this chapter we refer to [AAGM15, Pis99, BGVV14, TJ89].

1.1 Basics

By a *convex body* (or just a *body*) $K \subset \mathbb{R}^n$ we mean a compact convex set with non-empty interior. A body K is *centrally symmetric* if $x \in K$ implies $-x \in K$. Given a norm $\|\cdot\|$ on $\mathbb{R}^n$ we will write

$$B_X := \{x \in \mathbb{R}^n \mid \|x\| \leqslant 1\},$$

the unit ball of the space $X := (\mathbb{R}^n, \|\cdot\|)$. Unit balls are, obviously, centrally symmetric. In the other direction, we can define a norm on $\mathbb{R}^n$ associated to a centrally symmetric body by Minkowski's functional:

$$\|x\|_K := \inf\{\lambda > 0 \mid x \in \lambda K\}.$$

We write $X_K = (\mathbb{R}^n, \|\cdot\|_K)$ and we have that $B_{X_K} = K$. Hence, the study of centrally symmetric bodies corresponds to the study of different structures of $\mathbb{R}^n$ as a normed space. After fixing an Euclidean structure on $\mathbb{R}^n$ we can associate, to every convex body $K \subset \mathbb{R}^n$ with the origin as an interior point, its *polar body*,

$$K^\circ := \{y \in \mathbb{R}^n \mid \sup_{x \in K} \langle x, y \rangle \leqslant 1\}.$$

We need that the origin is an interior point of K in order for K° to be a bounded set. Note that, by definition, when K is centrally symmetric, K° is the unit ball of the dual space X_K^*. An important property of polarity is that it reverses inclusions, namely if $L \subset K$, $K^\circ \subset L^\circ$. It also also follows directly from the definition that, for an invertible linear operator T,

$$(T(K))^\circ = (T^{-1})^*(K^\circ). \tag{1.1}$$

An important family of convex bodies are the *polytopes*, the *convex hull* of some points $v_1, \dots, v_k$, that is, the set

$$\mathrm{conv}\{v_1, \dots, v_k\} := \{\sum_{i=1}^k t_i v_i \mid t_i \geqslant 0 \text{ and } \sum_{i=1}^k t_i = 1\}.$$

We will write $\mathrm{absconv}\{v_1, \dots, v_k\}$ for the *absolute convex hull*, namely $\mathrm{conv}\{\pm v_1, \dots, \pm v_k\}$. We can easily compute the polar body of a polytope as follows.

Example 1.1.1. Let $K = \mathrm{conv}\{v_1 \dots v_k\}$, then

$$K^\circ = \{x \in \mathbb{R}^n \mid \langle x, \sum_{i=1}^k t_i v_i\rangle \leqslant 1 \text{ for all} \sum_{i=1}^k t_i = 1\}$$

and hence, $x \in K^\circ$ if and only if $\langle x, v_i\rangle \leqslant 1$ for all $1 \leqslant i \leqslant k$. In other words, K° is the intersection of the hyperplanes $P_i = \{x \in \mathbb{R}^n \mid \langle x, v_i\rangle \leqslant 1\}$.

Along this work we are going to analyse some geometrical parameters associated to convex bodies in $\mathbb{R}^n$. We are interested in estimating the asymptotic behaviour of these parameters as functions of the dimension of the ambient space rather than computing their exact value. For two sequences of real numbers a_n and b_n we write $a_n \preceq b_n$ when there is a constant $C > 0$ (independent of n) such that $a_n \leqslant C b_n$ for every $n \in \mathbb{N}$. We write $a_n \sim b_n$ if $a_n \preceq b_n$ and $b_n \preceq a_n$. Probably one of the most famous asymptotic formula of all is Stirling's approximation formula of $n!$. We will make use of it again and again. It states that:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n. \tag{1.2}$$

It can be also generalized to approximate the Gamma function,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dx,$$

as follows

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x.$$

We will focus on volumetrical properties of convex bodies. By the volume of $K \subset \mathbb{R}^n$ we mean its Lebesgue measure that we will denote by $|K|$.

Example 1.1.2 (Volume of the p-balls)**.** A direct application of Stirling's formula is estimating the asymptotic behaviour of the volumes of the unit balls of the spaces $\ell_p^n = (\mathbb{R}^n, \|\cdot\|_p)$ that we write as B_p^n. A standard computation (see for example [Pis99, equation (1.17)]) shows that

$$|B_p^n| = \frac{\left(2\Gamma(1+\frac{1}{p})\right)^n}{\Gamma(1+\frac{n}{p})},$$

which by Stirling's formula behaves as

$$|B_p^n|^{\frac{1}{n}} \sim n^{-\frac{1}{p}}.$$

1.2 Volume inequalities

We now state some inequalities concerning the volume of convex bodies. The first one is the well known Blaschke-Santaló inequality that bounds the product between the volume of a convex body and the volume of its polar. This quantity is known as the *Mahler product* of K and, by equation (1.1), it is invariant under affine transformations. The next theorem shows that, in the case of centrally symmetric convex bodies, the Mahler product is maximized for ellipsoids. Balschke [Bla17] gave a proof for $n = 3$ and Santaló [San49] proved the general case. A simple proof using Steiner symmetrization was given by Meyer and Pajor in [MP90].

Theorem 1.2.1 (Blaschke-Santaló inequality)**.** *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body then,*

$$|K||K^\circ| \leqslant |B_2^n|^2.$$

Mahler conjectured that, among all convex bodies with the origin as an interior point, the simplex minimizes the Mahler product. He proved it for $n = 2$ [Mah39]. The following theorem is a reverse form of the last one, and shows that an asymptotic version of the Mahler conjecture holds. It was given by Bourgain and Milman [BM87] and a simplified proof can be found in [AAGM15, Theorem 8.2.2].

Theorem 1.2.2. *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body then,*

$$(|K||K^\circ|)^{\frac{1}{n}} \geq |B_2^n|^{\frac{2}{n}}.$$

The next inequalities are due to Rogers and Shephard [RS57] (also see [AAGM15, Theorem 1.5.2]). They will allow us to reduce many problems to the case of centrally symmetric bodies. Given K, the *difference body* is defined as the centrally symmetric convex body $D(K) = K - K$. It is the smallest centrally symmetric body that contains K. On the other hand, the biggest centrally symmetric convex body contained in K is $K \cap (-K)$. With $\mathrm{bar}(K)$ we mean the *barycenter* (or centroid) of K that is defined as,

$$\mathrm{bar}(K) = \frac{1}{|K|} \int_K x\,dx = 0.$$

The next theorem asserts that, in the case that $\mathrm{bar}(K) = 0$, all three bodies have comparable volume.

Theorem 1.2.3. *Let $K \subset \mathbb{R}^n$ a convex body then,*

$$|K - K| \leqslant \binom{2n}{n} |K|.$$

In case that $bar(K) = 0$ we have that,

$$|K \cap (-K)| \geqslant 2^{-n} |K|.$$

1.3 Some special positions

We say that K' is a *position* of K if there is an invertible affine transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(K) = K'$. We now present some classical positions of convex bodies.

1.3.1 John's and Löwner's positions

One of the most classical positions of convex bodies are John's and Löwner's positions. A convex body $K \subset \mathbb{R}^n$ is said to be in *John's position* if the Euclidean ball is the maximal volume ellipsoid inside K and is said to be in *Löwner's position* if B_2^n is the minimal volume ellipsoid enclosing K. Every convex body admits a unique (up to orthogonal transformations) John's or Löwner's position. The existence of such positions derives easily from a standard compactness argument. According to Busemann [Bus50, Bus55], Löwner discovered the uniqueness of the minimal volume ellipsoid but communicated the result orally. John, in [Joh48] extended the Lagrange multipliers rule to the case where the subsidiary conditions are inequalities (instead of equations). As a consequence he proved that, when B_2^n is the maximal volume ellipsoid inside K we must have

$$B_2^n \subset K \subset nB_2^n. \tag{1.3}$$

He also pointed out that when K is centrally symmetric, n can be replaced by $\sqrt{n}$, namely,

$$B_2^n \subset K \subset \sqrt{n} B_2^n. \tag{1.4}$$

Actually, John also gave a description of the contact points between K and B_2^n when K is in John's position. By a *contact point* between B_2^n and K we mean a point x, that lies in $\partial B_2^n \cap \partial K \cap \partial K^\circ$. Notice that, if $B_2 \subset K$ and $x \in \partial B_2^n \cap \partial K$, we must also have that $x \in \partial K^\circ$. We write $x_1 \otimes x_2$ for the rank one operator, $x_1 \otimes x_2(y) = \langle x_1, y \rangle x_2$. In the case that x has norm one, $x \otimes x$ is the orthogolan projection on the line generated by x. The next theorem is due to John and was complemented by Ball [Bal92].

Theorem 1.3.1. *Let $K \subset \mathbb{R}^n$ be a convex body. The Euclidean ball B_2^n is the ellipsoid of maximal volume inside K if and only if $B_2^n \subset K$ and there exist contact points $(x_j)_{j=1}^m$ and positive numbers $(c_j)_{j=1}^m$ such that*

$$\sum_{j=1}^m c_j x_j = 0$$

and

$$Id = \sum_{j=1}^m c_j x_j \otimes x_j. \tag{1.5}$$

We refer to (1.5) as a *decomposition of the identity.* It is not hard to check that any decomposition of the identity must fullfil that $\sum c_i = n$. Notice that, if $\mathcal{E} = T(B_2^n)$ is a centrally symmetric ellipsoid contained in K, $\mathcal{E}^\circ$ is an ellipsoid enclosing K°. And, by equation (1.1),

$$|\mathcal{E}||\mathcal{E}^\circ| = |B_2^n|^2.$$

Hence, if $\mathcal{E}$ is the centrally symmetric ellipsoid of maximal volume inside K, $\mathcal{E}^\circ$ must be the minimal volume ellipsoid enclosing K°. So, if K is in John's position, K° is Löwner's position. Since, by definition, the contact points in both cases are the same, we can also form a decomposition of the identity with those points if K is in Löwner's position. As an application of the decomposition of the identity we present a characterization of the John position in the case of a simplex.

Example 1.3.2. We say that a n-simplex, the convex hull of $n+1$ affinely independent points, is a *regular simplex* if all its vertices are equidistant. We will show that the simplex in Löwner's position is a regular one. Suppose that $S := \mathrm{conv}\{v_1, \ldots, v_{n+1}\}$ is in Löwner's position. Notice that, $n+1$ is the minimum amount of vertices that is needed to form a decomposition of the identity. Hence, there are positive numbers $(c_j)_{j=1}^{n+1}$ such that $Id = \sum_{j=1}^{n+1} c_j v_j \otimes v_j$. So, we have that

$$v_k = \sum_{j=1}^{n+1} c_j \langle v_j, v_k \rangle v_j = c_k v_k + \sum_{j \neq k} c_j \langle v_j, v_k \rangle v_j. \tag{1.6}$$

On the other hand, since $\sum_{j=1}^{n+1} c_j v_j = 0$,

$$v_k = -\frac{1}{c_k} \sum_{j \neq k} c_j v_j. \tag{1.7}$$

Combining (1.6) and (1.7) we get that,

$$\begin{aligned}
0 &= (c_k - 1) \left(-\frac{1}{c_k} \sum_{j \neq k} c_j v_j \right) + \sum_{j \neq k} c_j \langle v_j, v_k \rangle v_j \\
0 &= \sum_{j \neq k} c_j \left(\langle v_j, v_k \rangle - \frac{c_k - 1}{c_k} \right) v_j.
\end{aligned}$$

Since any choice of n vertices must be linearly independent, we have,

$$\langle v_j, v_k \rangle = \frac{c_k - 1}{c_k},$$

for every $1 \leqslant j, k \leqslant n + 1$. Which implies that all c_k are equal, and then $c_k = \frac{n}{n+1}$, and that the angle between all vertices is the same and hence are all equidistant. The polar of S is a simplex in John's position with faces given by $F_k = \{x \in \mathbb{R}^n \mid \langle x, v_k \rangle = 1\}$. Its vertices can be computed as

$$w_k = \bigcap_{i \neq k} F_i.$$

Hence,

$$\begin{aligned}
w_k &= \frac{n}{n+1} \sum_{i=1}^{n+1} \langle w_k, v_i \rangle v_i = \frac{n}{n+1} \sum_{i \neq k} v_i + \langle w_k, v_k \rangle v_k \\
&= \frac{n}{n+1} (-v_k + \langle w_k, v_k \rangle v_k) = \frac{n}{n+1} (\langle w_k, v_k \rangle - 1) v_k. \qquad (1.8)
\end{aligned}$$

Then,

$$\langle w_k, v_k \rangle = \frac{n}{n+1} (\langle w_k, v_k \rangle - 1) \langle v_k, v_k \rangle.$$

So we have,

$$\langle w_k, v_k \rangle = -n.$$

By (1.8) we deduce that $S^\circ = -nS$.

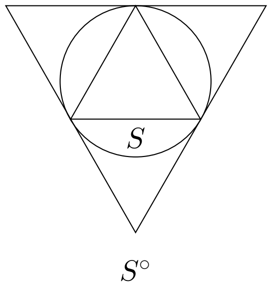

Figure 1.1: A regular simplex and its poplar body as in Example 1.3.2

1.3.2 Isotropic position

Another useful position arises from classical mechanics and is called the *isotropic position.* A convex body is said to be in isotropic position (or simply, is isotropic) if it has volume one and satisfies the following two conditions:

- $\mathrm{bar}(K) = 0$,
- $\displaystyle\int_K \langle x, \theta \rangle^2 \, dx = L_K^2 \quad \forall \theta \in S^{n-1}$,

where L_K is a constant independent of θ, which is called the *isotropic constant* of K. Notice that, the isotropic position can be understood in terms of the uniform measure on K. If K is isotropic, it is a probability measure with mean zero and covariance matrix which is a multiple of the identity. It is well known that every convex body admits a unique (up to orthogonal transformations) isotropic position (see for example [BGVV14, Proposition 2.3.3.]). Hence, we can define the isotropic constant of a convex body K as the isotropic constant of the isotropic affine image of K. The following proposition shows that L_K is always bounded from below. Its proof is quite simple and can be found, for example, in [AAGM15, Proposition 10.1.8].

Proposition 1.3.3. *For every convex body $K \subset \mathbb{R}^n$,*

$$L_K \geqslant L_{B_2^n} \succeq 1.$$

Its worth mentioning that it is unknown whether the isotropic constant is bounded from above by an absolute constant. This is maybe one of the main open questions in the area and has many equivalent formulations. The origin of this question is the so-called *slicing problem* (or *hyperplane conjecture*), which asks if every centered convex body of volume 1 has a hyperplane section through the origin whose volume is greater than an absolute constant $c > 0$. The hyperplane conjecture appears for the first time in the work of Bourgain [Bou86], but was stated in this form in an article of Milman and

Pajor [MP89], where the equivalence between many forms of the conjecture is proved. The best known general upper bound is $L_K \leqslant cn^{\frac{1}{4}}$, which was given by Klartag [Kla06] and improves the earlier estimate $L_K \leqslant cn^{\frac{1}{4}} \log n$ due to Bourgain [Bou91]. We now state two properties of isotropic bodies that we are going to use later. The first one is a well-known result of Kannan, Lovász and Simonovits [KLS95, Theorem 4.1.], it asserts that isotropic bodies contain a "large" unit ball.

Lemma 1.3.4. *Let $K \subset \mathbb{R}^n$ be an isotropic convex body, then*

$$\sqrt{\frac{n+2}{n}} L_K B_2^n \subset K. \tag{1.9}$$

The second one is related with a well behaviour of the marginals. Before stating it we need a definition. Let (Ω, Σ, μ) be a probability space and $f : \Omega \to \mathbb{R}$ a measurable function. We say that f is ψ_1 if there is $\lambda > 0$ such that

$$\int_\Omega e^{\frac{|f(\omega)|}{\lambda}} d\mu < \infty.$$

In that case the ψ_1-norm (or subexponencial norm) is defined as follows,

$$\|f\|_{\psi_1} := \inf\{\lambda > 0 \mid \int_\Omega e^{\frac{|f(\omega)|}{\lambda}} d\mu \leqslant 2\}.$$

It is worth mentioning that the ψ_1 norm is a particular case of an Orlicz-norm. For more information on this general framework see for example [AAGM15, Subsection 3.5.2] The next lemma gives a bound for the ψ_1 norm of the marginal of an isotropic body. A proof of it can be found in [BGVV14, Proposition 3.1.2].

Lemma 1.3.5. *There is an absolute constant $C > 0$ such that for every isotropic convex body $K \subset \mathbb{R}^n$ and every $\theta \in S^{n-1}$ we have*

$$\|\langle \cdot, \theta \rangle\|_{L_{\psi_1}} \leqslant C L_K.$$

1.3.3 ℓ-position

Now we introduce another special position that we are going to make use of. First we need some definitions. Given a centrally symmetric convex body K the *mean width* of K is defined as

$$\omega(K) = \int_{S^{n-1}} \|\theta\|_{K^\circ} d\sigma(\theta),$$

where σ stands for the normalized Haar's measure on S^{n-1}. The Spherical mean of a norm (actually of any homogeneous function) is related with its

Gaussian mean. If G is a vector with Gaussian independent coordinates, $\frac{G}{\|G\|_2}$ is uniformly distributed on the sphere, moreover $\|G\|_2$ and $\frac{G}{\|G\|_2}$ are independent. Hence, the spherical and Gaussian mean differ by the factor $\mathbb{E}(\|G\|_2)$. A standard computation together with Stirling's formula shows that,

$$\mathbb{E}(\|G\|_2) = \frac{\sqrt{2}\Gamma(\frac{n+1}{2}}{)}\Gamma(\frac{n}{2}) \sim \sqrt{n}.$$

Hence, if γ_n is the standard Gaussian measure on $\mathbb{R}^n$, the following holds

$$\sqrt{n}\int_{S^{n-1}} \|\theta\|_{K^\circ} d\sigma(\theta) \sim \int_{\mathbb{R}^n} \|x\|_{K^\circ} d\gamma_n(x). \tag{1.10}$$

Given a convex body K we define,

$$\ell(K) := \int_{\mathbb{R}^n} \|x\|_K d\gamma_n(x).$$

With this definition, equation (1.10) takes the following form

$$\ell(K) \sim \sqrt{n}\omega(K^\circ). \tag{1.11}$$

The more usual definition of the parameter ℓ is using the second moment, namely

$$\left(\int_{\mathbb{R}^n} \|x\|_K^2 d\gamma_n(x)\right)^{\frac{1}{2}}$$

instead of the first moment. Applying a Kahane-Khinchine type inequality for seminorms (see for example [TJ89, equation (4.5)]) we can see that

$$\left(\int_{\mathbb{R}^n} \|x\|_K^2 d\gamma_n(x)\right)^{\frac{1}{2}} \sim \int_{\mathbb{R}^n} \|x\|_K d\gamma_n(x). \tag{1.12}$$

Hence, from an asymptotic point of view both definitions of ℓ are equivalent. We use the first moment to make the relation with the mean width more transparent. The parameter ℓ was introduced by Figiel and Tomczak-Jaegermann in [FTJ79] in the context of operators norm. They proved that every convex body $K \subset \mathbb{R}^n$ admits a position $\bar{K}$ such that

$$\ell(\bar{K})\ell(\bar{K}^\circ) \leq n\,\mathrm{Rad}(\bar{K}),$$

where $\mathrm{Rad}(K)$ stands for the norm of the Rademacher projection $R_n : L^2(X_K) \to L^2(X_K)$. We will omit the proper definitions because we are not going to make use of them, we refer to [TJ89] and [Pis99] for more information on the subject. On the other hand, Pisier [Pis79] proved that for every n-dimensional Banach space X we have,

$$Rad(K) \leqslant c\log(d(K, B_2^n) + 1),$$

where $d(K, \ell_2^n)$ is the Banach-Mazur distance between K and B_2^n,

$$d(K, \ell_2^n) = \inf\{a \cdot b \mid \frac{1}{a}K \subset TB_2^n \subset bK\},$$

which, by John's theorem, is always bounded by $\sqrt{n}$. Putting all this together we obtain the following theorem.

Theorem 1.3.6. *Given a convex body $K \subset \mathbb{R}^n$ there is a position of K, $\tilde{K}$ such that $\ell(\tilde{K})\ell(\tilde{K}^\circ) \leq n \log(n+1)$.*

We end this chapter with a classical inequality of Urysohn, which relates the mean with of a convex body with its volume. Several proofs of it can be found in [AAGM15, Theorem 1.5.11].

Lemma 1.3.7 (Urysohn)**.** *Let $K \subset \mathbb{R}^n$ be a convex body. Then,*

$$\omega(K) \geqslant \left(\frac{|K|}{|B_2^n|}\right)^{\frac{1}{n}}.$$

Applying Stirling's formula, equation (1.11), and Bourgain-Milman inequality Theorem 1.2.2, Urysohn's inequality takes the following form,

$$\ell(K) \geq \frac{1}{|K|^{\frac{1}{n}}}. \tag{1.13}$$

Chapter 2

The simplex ratio

In this chapter we treat the problem of approximating a given convex body in $\mathbb{R}^n$ by an n-dimensional simplex of similar volume. This a generalization of an old geometrical question related to finding triangles of minimal area enclosing a given planar convex set. In Section 2.2 we review the rich history behind this problem. We deal with two dual versions of it, namely, approximating a given convex body $K \subset \mathbb{R}^n$ by a simplex contained in it or enclosing it. For this two formulations to be equivalent we need an additional property of the simplex: that has the same barycenter as K. We formulate the problem and prove the equivalence between both versions in Section 2.3. In Section 2.4 we use the probabilistic method to prove the existence of simplices which fulfill the desired properties. This gives our main result, Theorem 2.4.6, which consist on a probabilistic algorithm to find such simplices. Finally in Section 2.5 we present a non-probabilistic proof of the problem using a classical construction from Dvoretzky and Rogers regarding parallelepipeds enclosing convex bodies. While their construction proves the existence of a parallelepiped with the desired properties, it can be hard to find it explicitly. We also present a random version of the same result that allows us to avoid this problem.

2.1 Introduction

By a *simplex* $S \subset \mathbb{R}^n$ we always mean an n-dimensional simplex, the convex hull of $n+1$ affinely independent points. Given a convex body $K \subset \mathbb{R}^n$, we define the *outer simplex ratio* of K,

$$S^{out}(K) := \min \left(\frac{|S|}{|K|} \right)^{1/n},$$

where the minimum is taken over all simplices S in $\mathbb{R}^n$ containing K. Our goal is to give an asymptotic bound for the outer simplex ratio of a general convex body K. First notice that, since all simplices are in the same affine class, S^{out} happens to be an affine invariant quantity of K. In fact,

$$\begin{aligned} S^{out}(TK) = \min_{TK\subset S}\left(\frac{|S|}{|TK|}\right)^{1/n} &= \min_{TK\subset S}\left(\frac{|T^{-1}(S)|}{|K|}\right)^{1/n} \\ &= \min_{K\subset \tilde{S}}\left(\frac{|\tilde{S}|}{|K|}\right)^{1/n}. \end{aligned}$$

Let's start with a few examples to illustrate the problem.

Example 2.1.1 (The minimal volume simplex for the Euclidean ball)**.** Let $K := B_2^n$, the Euclidean ball, and $S \supset B_2^n$ the regular simplex defined in Example 1.3.2.

Let us see that S is the minimal volume simplex containing K. If not, then there is a simplex $T \subset \mathbb{R}^n$ enclosing the ball with $\mathrm{vol}(T) < \mathrm{vol}(S)$. Consider the linear transformation $A \in GL(n)$ such that $A(S) = T$; then, $|\det(A)| < 1$. Therefore $A^{-1}(B_2^n)$ is an ellipsoid with volume greater than $|B_2^n|$ inside S, which contradicts the fact that S is in John's position.

In order to compute the volume of the simplex it is easier to do it working in $\mathbb{R}^{n+1}$ on the hyperplane $\sum_i x_i = 1$. There the simplex is represented as the n-dimensional simplex $\mathrm{conv}\{e_1, \ldots, e_{n+1}\}$, that contains an n-dimensional sphere centered in the point $(\frac{1}{n+1}, \ldots, \frac{1}{n+1})$ and has radius,

$$r = \left\|(\frac{1}{n+1}, \ldots, \frac{1}{n+1}) - (\frac{1}{n}, \ldots, \frac{1}{n}, 0)\right\| = \frac{1}{\sqrt{n(n+1)}}. \tag{2.1}$$

Notice that, by this identification, S is a face of the simplex $\Delta := \mathrm{conv}\{0, e_1, \ldots, e_{n+1}\}$ that has volume $\frac{1}{(n+1)!}$. If we think the simplex as a cone with base S, the volume can be also computed as

$$|\Delta| = \frac{1}{n+1}|S|_n\|(\frac{1}{n+1}, \ldots, \frac{1}{n+1})\| = \frac{1}{n+1}|S|_n\frac{1}{\sqrt{n+1}}.$$

We now rescale by the radius computed in equation (2.1) and obtain:

$$|S| = \frac{(n+1)^{\frac{n+1}{2}} n^{\frac{n}{2}}}{n!}. \tag{2.2}$$

On the other hand, the volume of the unit ball was computed in Example 1.1.2,

$$|K| = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}.$$

Using Stirling formula (equation (1.2)) we therefore get

$$\left(\frac{|S|}{|K|}\right)^{\frac{1}{n}} \sim \sqrt{n}.$$

We conclude that $S^{out}(B_2^n) \sim \sqrt{n}$.

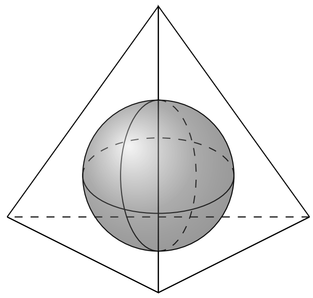

Figure 2.1: The simplex of minimal volume enclosing the Euclidean ball is the regular simplex circumscribing it.

Example 2.1.2 (The cube)**.** Let K be the cube $[0,1]^n \subset \mathbb{R}^n$ and consider the simplex $S := \operatorname{conv}\{0, ne_1, \ldots, ne_n\}$. We have that $K \subset S$ and

$$\left(\frac{|S|}{|K|}\right)^{\frac{1}{n}} \sim 1. \tag{2.3}$$

In fact, since the sum of the coordinates of all points that lie in the cube is less than or equal to n, we have that $K \subset S$. The volume of the cube is 1, and the volume of the simplex is $\frac{\det|ne_1,\ldots,ne_n|}{n!} = \frac{n^n}{n!}$. Applying Stirling formula,

$$\left(\frac{|S|}{|K|}\right)^{\frac{1}{n}} = \left(\frac{n^n}{n!}\right)^{\frac{1}{n}} \sim 1.$$

Since $S^{out}(K) \geqslant 1$ always holds, we have that $S^{out}(K) \sim 1$.

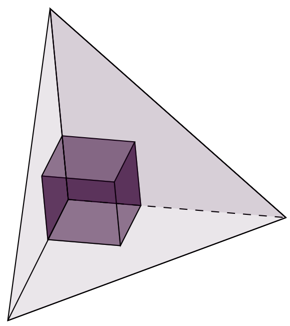

Figure 2.2: The simplex $\operatorname{conv}\{0, 3e_1, 3e_2, 3e_3\}$ enclosing the cube $[0,1]^3 \subset \mathbb{R}^3$ as in Example 2.1.2.

2.2 Historical background

Before stating our main results we are going to review the history behind this problem. For the Euclidean plane, i.e. $n = 2$, it was completely solved by Gross [Gro18] (and generalized in different ways by W. Kuperberg [Kup83]): every convex body $K \subset \mathbb{R}^2$ can be inscribed in a triangle of area at most $2|K|$ (see Figure 2.3). This ratio corresponds (exclusively) to the case that K is a parallelogram. The measure of the tetrahedron (not necessarily regular) of least volume enclosing a convex body $K \subset \mathbb{R}^3$ is in general unknown. If $K \subset \mathbb{R}^3$ is a parallelepiped of volume one, then the minimal volume tetrahedron containing it has volume $9/2$. It is an open question whether this is the worst possible fit for the general case. To our knowledge, there are not even conjectured bounds for greater dimensions ($n \geqslant 4$).

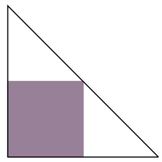

Figure 2.3: Triangle of minimal area enclosing the unit square

An n-dimensional bound for the simplex ratio was given by Macbeath in [Mac51a] where he constructs a simplex enclosing a convex body K such that $|S| \leqslant n^n|K|$, what implies the bound $S^{out}(K) \leqslant n$. This bound was improved (but with the same asymptotic order) in the seventies by Chakerian [Cha73, Corollary 5]. The same estimate was recently rediscovered in 2014 by Kanazawa [Kan14, Theorem 1] using different arguments. In particular, both authors showed that

$$S^{out}(K) \leqslant n^{\frac{n-1}{n}} \approx n. \tag{2.4}$$

Note that when $n = 2$ this is just Gross' bound.

It is possible to improve the previous bound applying a general inequality for volume ratios due to Giannopoulos and Hartzoulaki [GH02], where they reduced the problem to the centrally symmetric case and applied Chevet's inequality (Theorem 3.3.2) for a position of the bodies related with the ℓ-position (Theorem 1.3.6). We will discuss their construction more deeply in Chapter 3. As a consequence of their results we have

$$S^{out}(K) \preceq \sqrt{n}\log(n). \tag{2.5}$$

Up to our knowledge, this is the best known bound so far; see also the recent work of Paouris and Pivovarov [PP17, Corollary 5.4], regarding a randomized version of Urysohn inequality, where the same bound is given but with a different method.

By duality, bounding the outer simplex ratio is related with finding simplices of large volume *inside* a convex body with the same barycenter. The search of simplices of large volume contained in a convex body has an extensive and interesting history in geometry. For instance, the study of the maximum area of triangles in planar convex bodies was undertaken by Blaschke [Bla17] in the early 20th century. Sas [Sas39] and Macbeath [Mac51b] also considered the problem of approximating a given convex body by inscribed polytopes. McKinney [McK74] studied certain properties of those simplices of maximum volume inside a centrally symmetric convex body. The survey [HKL96] also deals with simplices of large volume in cubes.

A classical question regarding simplices inside convex bodies was stated by Sylvester [Syl65]: Given 4 points uniformly distributed on a planar convex body K, what is the probabilty of its convex hull being a simplex? This is directly related to estimating the expected volume of a random simplex with vertices taken uniformly on a convex body. Given $K \subset \mathbb{R}^n$ of volume 1, set

$$S_p(K) := \left(\int_K \cdots \int_K |\operatorname{conv}\{x_1 \dots x_{n+1}\}|^p dx_1 \dots dx_{n+1} \right)^{\frac{1}{p}}. \tag{2.6}$$

The so-called *Sylvester problem* is to describe the affine classes of convex bodies for which $S_p(K)$ is maximized or minimized. For $n = 2$ it was settled by Blashke [Bla17], the minimizers and maximizers are ellipsoids and simplices respectively.

Groemer [Gro73] proved that $S_p(K) \geqslant S_p(B_2^n)$ (Blashke-Groemer inequality) holds for every convex body K and equality is attained if and only if K is an ellipsoid. In the opposite direction the problem is open for $n \geqslant 3$. It is conjectured that the maximizer is the simplex, this is known as the *simplex conjecture*. Milman and Pajor [MP89] established a relation between $S_1(K)$ and the isotropic constant of K, they proved that

$$S_1(K) \sim \frac{L_k}{\sqrt{n}}. \tag{2.7}$$

As consequence of this relation we have that the simplex conjecture implies an affirmative answer to the slicing problem.

Busemann [Bus53] introduced the following variant of Sylvester's functional,

$$B_p(K) := \left(\int_K \cdots \int_K |\operatorname{conv}\{0, x_1, \dots, x_n\}|^p dx_1 \dots dx_n \right)^{\frac{1}{p}}. \tag{2.8}$$

Again, $B_p(K)$ is minimal among all convex bodies when K is an ellipsoid. That leads to the next inequality, known as Busemann's random simplex inequality [Bus53].

$$B_1(K) \geqslant \left(\frac{|B_2^{n-1}|}{(n+1)|B_2^n|} \right)^n |K|^{n+1} \geqslant \left(\frac{c}{\sqrt{n}} \right)^n |K|^{n+1}, \tag{2.9}$$

with $c > 0$ an absolute constant.

2.3 Inner simplex ratio and duality

In order to obtain a general bound for $S^{out}(K)$ we are going to approach a dual version of the problem: finding simplices of large volume contained in K.

Given a convex body $K \subset \mathbb{R}^n$ we define the *inner simplex ratio* of K as

$$S^{inn}(K) := \min \left(\frac{|K|}{|S|} \right)^{1/n},$$

where the minimum is taken over all simplices $S \subset K$.

Example 2.3.1. Let $K := B_2^n$ and S the regular simplex in Löwner's position. The same argument that was used in Example 2.1.1 shows that S is the maximal volume simplex inside K. Since S can be obtained stretching the regular simplex that enclose the ball by a factor of n, from equation (2.2) we have that,

$$|S| = \frac{(n+1)^{\frac{n+1}{2}} n^{\frac{n}{2}}}{n! n^n} \sim \frac{1}{n}, \tag{2.10}$$

and hence $S^{inn}(B_2^n) \sim \sqrt{n}$.

There are many ways of showing that for every body K, $S^{inn}(K) \preceq \sqrt{n}$. It can be found in the work of Macbeath in [Mac51a]. It was also showed by Giannopoulos, Perissinaki and Tsolomitis in [GPT01]. We will show how to prove it using the classic Dvoretzky-Rogers, Lemma that asserts that it is possible to extract from a decomposition of the identity an "almost" orthogonal basis.

Lemma 2.3.2 (Dvoretzky-Rogers)**.** *Let $w_1, \dots, w_m \in S^{n-1}$ and $c_1, \dots, c_m$ such that $Id = \sum c_i w_i \otimes w_i$, then there is a subset of vectors $\{v_1, \dots, v_n\} \subset \{w_1, \dots, w_m\}$ such that $\|P_k(v_{k+1})\|_2 \geqslant \left(\frac{n-k}{n}\right)^{\frac{1}{2}}$ for $1 \leqslant k \leqslant n-1$, where P_k stands for the orthogonal projection on $span\{v_1, \dots, v_k\}^{\perp}$.*

Proof. First notice that for a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ there must be w_i such that $\langle w_i, T w_i \rangle \geqslant \frac{\text{tr}(T)}{n}$. In fact,

$$\frac{\text{tr}(T)}{n} = \frac{1}{n} \sum_{i=1}^{n} c_i \langle T, w_i \otimes w_i \rangle = \frac{1}{n} \sum_{i=1}^{n} c_i \langle T w_i, w_i \rangle,$$

and for some vector w_i, $\langle T w_i, w_i \rangle$ must be greater than the arithmetic mean. The proof goes by induction. Set $v_1 = w_1$ and suppose we already have

$v_1 \dots v_k$. If P_k is the orthogonal projection on $\mathrm{span}\{v_1 \dots v_k\}^\perp$, $\mathrm{tr}(P_k) = n-k$. Take w_i such that $\langle w_i, P_k w_i\rangle \geqslant \frac{n-k}{n}$, we have

$$\|P_k(w_i)\|_2 = \langle w_i, P_k w_i\rangle^{\frac{1}{2}} \geqslant \left(\frac{n-k}{n}\right)^{\frac{1}{2}},$$

which concludes the proof. □

Now, consider $\{v_1, \dots, v_n\}$ given by Dvoretzky-Rogers lemma and define the simplex $S := \mathrm{conv}\{0, v_1, \dots, v_n\}$. We can think of this simplex as a cone with base $\mathrm{conv}\{0, v_1, \dots, v_{n-1}\}$. Its volume can be computed as

$$\frac{1}{n}\|P_{n-1}(v_n)\|_2 |\mathrm{conv}\{0, v_1, \dots, v_{n-1}\}|_{n-1}.$$

Since $\mathrm{conv}\{0, v_1, \dots, v_{n-1}\}$ is itself a cone with base $\mathrm{conv}\{0, v_1, \dots, v_{n-2}\}$, we iterate the argument and get

$$|S| = \frac{1}{n!}\|P_{n-1}(v_n)\|_2 \|P_{n-2}(v_{n-1})\|_2 \dots \|v_1\|_2 \geqslant \frac{\sqrt{(n-1)!}}{n!\sqrt{n!}} \sim \frac{1}{n}. \tag{2.11}$$

As a consequence of this we have the following proposition regarding the inner simplex ratio of a general convex body.

Proposition 2.3.3. *Given a convex body $K \subset \mathbb{R}^n$ there is a simplex $S \subset K$ such that $\left(\frac{|K|}{|S|}\right)^{\frac{1}{n}} \preceq \sqrt{n}$.*

Proof. Assume that K is in Löwner's position and consider a decomposition of the identity formed by contact points between K and B_2^n as in Theorem 1.3.1. Extract from the contact points the vectors given by Lemma 2.3.2 and consider the simplex $S := \mathrm{conv}\{0, v_1, \dots, v_n\}$. Since $K \subset B_2^n$ we have

$$\left(\frac{|K|}{|S|}\right)^{\frac{1}{n}} \leqslant \left(\frac{|B_2^n|}{|S|}\right)^{\frac{1}{n}}.$$

The result follows from equation (2.11) and Stirling formula. □

A bound of the same asymptotic order can be obtained considering random simplices formed with vertices uniformly distributed inside K. Applying equations (2.8) or (2.7) we can prove the existence of simplices inside K with volume of the same order as the one showed before.

In order to deduce a bound for the outer ratio from the inner one it is necessary to require an additional property of the simplex, that it shares the same barycenter as the given convex body. The main reason for this is that we make use of polarity to pass from one to another. Given a convex body $K \subset \mathbb{R}^n$ and a simplex $S \subset K$ we have that $S^\circ \supset K^\circ$, and we need the Blaschke-Santaló inequality (Theorem 1.2.1) in order to relate the volume of S and S°. For a non centrally symmetric convex body L, the Blaschke-Santaló inequality takes the following form:

$$\min_{x\in L} |L||(L-x)^\circ| \leqslant |B_2^n|^2.$$

The point where the minimun is attained is called the *Santaló point* of L. Hence, we can only relate the volume of L with the volume of the polar body of a translation of L. So, in order for the inclusion between S and K to hold, we would need to move K as well, but in that case we loose control of the volume of the polar body of K. It is known, see for example [Sch14, equation 10.23], that if $\mathrm{bar}(L^0) = 0$ then the Santaló point of L is at the origin, and that is the case of a centered simplex, as we will see in Lemma 2.3.7. This induces a stronger version of the aforementioned problem. Given a convex body $K \subset \mathbb{R}^n$, we define

$$S_\circ^{out}(K) := \min \left(\frac{|S|}{|K|}\right)^{\frac{1}{n}},$$

where the minimum is taken over all simplices S containing K *and having the same barycenter.*

And similarly,

$$S_\circ^{inn}(K) := \min \left(\frac{|S|}{|K|}\right)^{\frac{1}{n}},$$

where the minimum is taken over all simplices S included in K *and having the same barycenter.* The problem now is to obtain general bounds for both, $S_\circ^{out}(K)$ and $S_\circ^{inn}(K)$.

Theorem 2.3.4. *Given a convex body $K \subset \mathbb{R}^n$ there is a simplex S with the same barycenter such that $K \subset S$ and $\left(\frac{|S|}{|K|}\right)^{\frac{1}{n}} \leq \sqrt{n}$.*

Theorem 2.3.5 (Dual version)**.** *Given a convex body $K \subset \mathbb{R}^n$ there is a simplex S with the same barycenter such that $S \subset K$ and $\left(\frac{|K|}{|S|}\right)^{\frac{1}{n}} \leq \sqrt{n}$.*

Before proving them we will show how to deduce Theorem 2.3.4 from Theorem 2.3.5. We start with two lemmas regarding the barycenter of a simplex and its polar body.

Lemma 2.3.6. *The barycenter of a simplex $S = \mathrm{conv}\{v_1, \dots, v_{n+1}\}$ is given by the arithmetic mean of it vertices,*

$$bar(S) = \frac{1}{n+1}\sum_{i=1}^{n+1} v_i.$$

Proof. First observe that if $f : \mathbb{R}^n \to \mathbb{R}^n$ is an affine transformation, $f(x) = Ax + v$, and $K \subset \mathbb{R}^n$ is a convex body we have that $f(\mathrm{bar}(K)) = \mathrm{bar}(f(K))$, in fact,

$$\begin{aligned}\mathrm{bar}(f(K))_i &= \frac{1}{|f(K)|}\int_{f(K)} x_i dx_1 \dots dx_n \\ &= \frac{1}{\det(A)|K|}\int_K f(x)_i \det(A) dx_1 \dots dx_n = \mathrm{bar}(f(K)).\end{aligned}$$

Hence, since any simplex is the affine image of $\Delta = \mathrm{conv}\{0, e_1, \dots, e_n\}$ we only need to prove the assertion for this simplex. Notice that Δ is invariant under permutation of coordinates and so must be its barycenter. So, we have that $\mathrm{bar}(\Delta) = (b, \dots, b)$ and the only point like this that lies in Δ is $(\frac{1}{n+1}, \dots, \frac{1}{n+1})$. $\square$

Lemma 2.3.7. *Let $S \subset \mathbb{R}^n$ be a simplex with barycenter at origin, then S° is also a centered simplex.*

Proof. Let $\Delta_0 := conv\{e_1, \dots, e_n, -\sum e_i\}$. From the previous lemma we know that the barycenter of a simplex $\mathrm{conv}\{v_0, \dots, v_n\}$ is given by the arithmetic mean of the vertices,

$$\mathrm{bar}(\mathrm{conv}\{v_0, \dots, v_n\}) = \frac{1}{(n+1)}\sum_{i=0}^{n} v_i. \tag{2.12}$$

We know that there is an affine transformation T such that $S = T\Delta_0$. Since S and Δ_0 have barycenter at the origin, T must be linear. Since $S^\circ = (T^*)^{-1}\Delta_0^\circ$, to show that $\mathrm{bar}(S^\circ) = 0$ is enough to do it for Δ_0°.

The polar body of a polytope is the intersection of the hyperplanes determined by its vertices (Example 1.1.1), we have that $\Delta_0^\circ = \bigcap\{x : \langle x, e_i\rangle \leqslant 1\}\bigcap\{x : \langle x, -\sum e_i\rangle \leqslant 1\}$ and its $n+1$ vertices are given by all possible intersection of n of these hyperplanes. So, $\Delta_0 = \{v_1, \dots, v_{n+1}\}$ with

$$v_{n+1} = \bigcap_{i=1}^{n}\{x : \langle x, e_i\rangle = 1\} = \sum_{i=1}^{n} e_i$$

and

$$v_k = \bigcap_{i\neq k}\{x : \langle x, e_i\rangle = 1\}\bigcap\{x : \langle x, -\sum_{i=1}^{n} e_i\rangle = 1\} = (1, 1, \dots, -n, \dots, 1).$$

The result follows from the fact that $\sum_{i=1}^{n+1} v_i = 0$. $\square$

Proof of Theorem 2.3.4 assuming Theorem 2.3.5. Let $K \subset \mathbb{R}^n$ be an arbitrary convex set with barycenter at the origin. By the Rogers-Shephard inequality, Theorem 1.2.3, the centrally symmetric difference body $D(K) = K - K$ contains K and fulfills

$$\left(\frac{|D(K)|}{|K|}\right)^{1/n} \leqslant 4. \tag{2.13}$$

By Theorem 2.3.5 applied to the body $D(K)^\circ$ there is a simplex with barycenter at the origin $T \subset D(K)^\circ$ such that

$$\left(\frac{|D(K)^\circ|}{|T|}\right)^{\frac{1}{n}} \preceq \sqrt{n}. \tag{2.14}$$

Consider S the simplex T°. By Lemma 2.3.7, S has also barycenter at the origin and obviously $S \supset D(K)$. Now,

$$\frac{|S|}{|D(K)|} = \frac{|S||T|}{|D(K)||D(K)^\circ|} \cdot \frac{|D(K)|^\circ}{|T|}. \tag{2.15}$$

By Blaschke Santaló, Bourgain-Milman inequalities and Stirling formula we have

$$\left(\frac{|S||T|}{|D(K)||D(K)^\circ|}\right)^{1/n} \preceq 1. \tag{2.16}$$

The result now follows immediately form equations (2.13), (2.14), (2.15), (2.16) and the fact that $D(K) \supset K$ and hence $S \supset K$. $\square$

Remark 2.3.8. In the previous proof we pass through the difference body of K because of the polar body of a centered body is not necessary centered (see [MSW10]).

Examples 2.1.1 and 2.3.1 show that the bounds obtained are asymptotically sharp.

2.4 A probabilistic approach

The probabilistic method is a standard method for proving the existence of a specified kind of mathematical object. The philosophy is to show that if one randomly chooses objects from a fixed class, the probability that the result is of the prescribed type is positive. In this section we use this method to give a proof of a stronger version of Theorem 2.3.5. For this we need two propositions that essentially state that, with very high probability, certain random simplices have "good properties".

Suppose $K \subset \mathbb{R}^n$ is an isotropic convex body and we randomly choose $X_1, \ldots, X_n$ in K. The following statement asserts that typically the barycenter of the random simplex $\operatorname{conv}\{0, X_1, \ldots, X_n\}$ has "small" norm.

Proposition 2.4.1. *There is an absolute constant* $c_1 > 0$ *such that for every isotropic convex body* $K \subset \mathbb{R}^n$ *and* $\{X_i\}_{i=1}^n$ *independent random vectors uniformly distributed in* K *then*

$$\mathbb{P}\{\|bar(T)\| \leqslant c_1 L_K\} > 1 - \frac{1}{2}e^{-n}, \tag{2.17}$$

where T *is the random simplex* $\mathrm{conv}\{0, X_1, \ldots, X_n\}$*.*

Our arguments to prove this proposition are based on the proofs of [AG08, Theorem 3.1.] and [KK09, Theorem 1.1.].

We need to state a technical lemma. Given a meric space M, a δ-*net* for M is a set $\mathcal{N} \subset M$ such that for every $x \in M$ there is $\eta \in \mathcal{N}$ such that $d(x, \eta) \leqslant \delta$.

Lemma 2.4.2. *Let* $\delta > 0$ *and* $n \in \mathbb{N}$*. There is a* δ*-net* $\mathcal{N}$ *for* S^{n-1} *with cardinality* $\#(\mathcal{N}) \leqslant (1 + \frac{2}{\delta})^n$*.*

Proof. The proof follows by a standard volumetric argument. Let $\{x_i\}$ be a maximal δ-separated set in S^{n-1}. Then $\{x_i\}_{i=1}^N$ is a δ-net for S^{n-1}. Since the sets $x_i + \frac{1}{2}\delta B_2^n$ are disjoint and are all included in $B_2^n + \frac{\delta}{2}B_2^n$, taking volume we get that

$$\#(\mathcal{N})(\frac{\delta}{2})^n|B_2^n| \leqslant (1 + \frac{\delta}{2})^n|B_2^n|, \tag{2.18}$$

which gives the desired bound. □

We also need a classical inequality due to Bernstein about sums of independent random variables (see, for example [AAGM15, Theorem 3.5.16]) together with the "good behavior" of the marginals $\langle \cdot, \theta \rangle$, for any direction $\theta \in S^{n-1}$ (Lemma 1.3.5).

Theorem 2.4.3 (Bernstein inequality)**.** *Let* $\{Y_i\}_{i=1}^n$ *be a sequence of random variables with mean* 0 *on some probability space. Assume that* Y_i *belong to* L_{ψ_1} *and that* $\|Y_i\|_{L_{\psi_1}} \leqslant M$ *for all* $i = 1, \ldots, n$*. Let* $\sigma^2 = \frac{1}{n}\sum_{i=1}^n \|Y_i\|_{L_{\psi_1}}^2$*. Then, for all* $t > 0$*,*

$$\mathbb{P}\left\{\left|\sum_{i=1}^n Y_i\right| > tn\right\} \leqslant e^{-D\, n \min\{\frac{t^2}{\sigma^2}, \frac{t}{M}\}}, \tag{2.19}$$

for some absolute constant $D > 0$*.*

We are now ready to give a proof of Proposition 2.4.1.

Proof of Proposition 2.4.1. Let $\{X_i\}_{i=1}^n$ be independent random vectors uniformly distributed on K and let θ be a fixed direction in S^{n-1}.

By combining Lemma 1.3.5 and Theorem 2.4.3 for the random variables $Y_j := \langle X_j, \theta \rangle$ we have, for all $t > CL_K$,

$$\mathbb{P}\left\{ \left| \langle \sum_{i=1}^{n} X_i, \theta \rangle \right| > tn \right\} \leqslant e^{-n \frac{t\,D}{C\,L_K}}.$$

Let $\mathcal{N}$ be a $\frac{1}{2}$-net on the sphere of cardinality less than or equal to 5^n given by Lemma 2.4.2. Then

$$\mathbb{P}\left\{ \left| \langle \sum_{i=1}^{n} X_i, \theta \rangle \right| > tn \text{ for some } \theta \in \mathcal{N} \right\} \leqslant e^{-n(\frac{t\,D}{C\,L_K} - \log(5))},$$

and hence

$$\mathbb{P}\left\{ \left| \langle \sum_{i=1}^{n} X_i, \theta \rangle \right| \leqslant tn \text{ for every } \theta \in \mathcal{N} \right\} \geqslant 1 - e^{-n(\frac{t\,D}{C\,L_K} - \log(5))}.$$

Every vector $\vartheta \in S^{n-1}$ can be written in the form $\vartheta = \sum_{j=1} \delta_j x_j$, with $x_j \in \mathcal{N}$ and $0 \leqslant \delta_j \leqslant 2^{1-j}$. In fact, start with $\curvearrowleft_1 \in \mathcal{N}$ such that $\|\vartheta - x_1\|_2 = \delta_1 \leqslant 1$. Then, $\frac{\vartheta - x_1}{\delta_1} \in S^{n-1}$ and hence there is $x_2 \in \mathcal{N}$ with $\|\frac{\vartheta - x_1}{\delta_1} - x_2\|_2 = \delta_2 \leqslant \frac{1}{2}$. So, we have

$$\|\vartheta - x_1 - \delta_1 x_2\|_2 \leqslant \delta_1 \delta_2.$$

Inductively we find $x_1, \ldots, x_n \in \mathcal{N}$ and $\delta_1, \ldots, \delta_n$ such that

$$\left\| \vartheta - \sum_{i=1}^{n} \left(\prod_{j=1}^{i} \delta_j \right) x_i \right\|_2 \leqslant 2^{j-1}.$$

Then $\vartheta = \sum_{i=1}^{\infty} (\prod_{j=1}^{i} \delta_j) x_i$. Observe that

$$\begin{aligned} \bigcap_{\theta \in \mathcal{N}} \left\{ \left| \langle \sum_{i=1}^{n} X_i, \theta \rangle \right| \leqslant tn \right\} &\subset \left\{ \left\| \sum_{i=1}^{n} X_i \right\|_2 \leqslant 2tn \right\} \\ &= \left\{ \max_{\vartheta \in S^{n-1}} \left| \langle \sum_{i=1}^{n} X_i, \vartheta \rangle \right| \leqslant 2tn \right\}. \end{aligned}$$

Indeed, let ϑ be an arbitrary unit vector and suppose that $|\langle \sum_{i=1}^{n} X_i, \theta \rangle| \leqslant tn$ for every $\theta \in \mathcal{N}$, then

$$\left| \langle \sum_{i=1}^{n} X_i, \vartheta \rangle \right| = \left| \langle \sum_{i=1}^{n} X_i, \sum_{j=1}^{\infty} \delta_j \theta_j \rangle \right| \leqslant \sum_{j=1}^{\infty} \delta_j \left| \langle \sum_{i=1}^{n} X_i, \theta_j \rangle \right| \leqslant 2tn.$$

Thus, for every $t > CL_K$ we have

$$\mathbb{P}\left\{ \left\| \sum_{i=1}^{n} X_i \right\|_2 \leqslant 2tn \right\} \geqslant 1 - e^{-n(\frac{t\,D}{C\,L_K} - \log(5))}.$$

The result now follows by setting $t := \frac{c_1(n+1)L_K}{2n}$, for $c_1 > 0$ sufficiently large. $\square$

The second proposition we need ensures that, typically, the random simplex $\operatorname{conv}\{0, X_1, \ldots, X_n\}$ has "large volume".

Proposition 2.4.4. *There is an absolute constant $c_2 > 0$ such that for every isotropic convex body $K \subset \mathbb{R}^n$ and $\{X_i\}_{i=1}^n$ independent random vectors uniformly distributed in K then*

$$\mathbb{P}\left\{|\operatorname{conv}\{0, X_1 \ldots, X_n\}| \geqslant \frac{c_2^n L_K^n}{n^{\frac{n}{2}}}\right\} > 1 - \frac{1}{2}e^{-n}. \tag{2.20}$$

A proof of it can be found essentially in the work of Pivovarov [Piv10, Proposition 1]. We include the details for completeness.

Lemma 2.4.5. *[Piv10, Lemma 2] Let $K \subset \mathbb{R}^n$ be an isotropic convex body and X be a random vector uniformly distributed on K. Let $E \subset \mathbb{R}^n$ be a k-dimensional subspace and P_E the orthogonal projection onto E. Then the random variable*

$$Y := \frac{\|P_E(X)\|_2}{L_K\sqrt{k}}$$

satisfies

$$\mathbb{E}\,|Y|^{-\frac{1}{2}} \leqslant C',$$

where $C' > 0$ is an absolute constant.

Proof of Proposition 2.4.4. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation mapping the canonical basis $\{e_i\}_{i=1}^n$ to $\{X_i\}_{i=1}^n$. In this case,

$$\operatorname{conv}\{0, X_1, \ldots, X_n\} = A\left(\operatorname{conv}\{0, e_1, \ldots, e_n\}\right)$$

and so we have

$$|\operatorname{conv}\{0, X_1, \ldots, X_n\}| = \frac{|\det(A)|}{n!}. \tag{2.21}$$

Set $V_k := \operatorname{span}\{X_1, \ldots, X_k\}$ and $Y_k = \frac{\|P_{V_{k-1}{}^\perp} X_k\|_2}{L_K\sqrt{n-k+1}}$. By Lemma 2.4.5 if $X_1, \ldots, X_{k-1}$ are fixed we have $\mathbb{E}[|Y_k|^{-\frac{1}{2}}] \leqslant C'$.

Computing the volume of $\operatorname{conv}\{0, X_1, \ldots, X_n\}$ as in equation (2.11),

$$|\det(A)| = \|X_1\|_2 \|P_{V_1{}^\perp(X_2)}\|_2 \ldots \|P_{V_{n-1}{}^\perp(X_n)}\|_2 \tag{2.22}$$

and applying Fubbini theorem iteratively we obtain

$$\mathbb{E}[\prod_i^n Y_k^{-\frac{1}{2}}] \leqslant (C')^n. \tag{2.23}$$

Let $\alpha > 0$ be a constant to be determined. Then by the Markov inequality and equation (2.23) we have

$$\begin{aligned}\mathbb{P}(|\det(A)| < \alpha^n L_K^n \sqrt{n!}) &= \mathbb{P}(\prod_i^n Y_k < \alpha^n) \\ &= \mathbb{P}(\prod_i^n Y_k^{-\frac{n}{2}} > \alpha^{-\frac{n}{2}}) \\ &\leqslant \mathbb{E}[\prod_i^n Y_k^{-\frac{1}{2}}]\alpha^{\frac{n}{2}}.\end{aligned}$$

Setting $\alpha = (eC')^{-2}$ we obtain

$$\mathbb{P}(|\operatorname{conv}\{0, X_1, \dots, X_n\}| < \frac{\alpha^n L_K^n}{\sqrt{n!}}) \leqslant \frac{1}{2}e^{-n}.$$

The result follows by applying Stirling formula. □

The next theorem is a stronger version of Theorem 2.3.5. If the convex body K is in isotropic position, it gives a probabilistic method to find simplices inside K (having barycenter at the origin) with volume large enough. We believe this result is interesting in its own right. Here $\mathcal{S}_0^n$ stands for the set of centered simplices in $\mathbb{R}^n$.

Theorem 2.4.6. *There exists a function* $f_n : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n} \to \mathcal{S}_0^n$ *such that for every isotropic convex body* $K \subset \mathbb{R}^n$ *and* $X_1, \dots, X_n$ *independent random vectors uniformly distributed on* K*, with probability greater than* $1 - e^{-n}$ *we have that* $f_n(X_1, \dots, X_n)$ *is a simplex with barycenter at the origin contained in* K *such that*

$$|f_n(X_1, \dots, X_n)| \geqslant \frac{c^n L_K^n}{n^{n/2}}, \tag{2.24}$$

where $c > 0$ *is an absolute constant.*

We base the following arguments on the recent paper of Naszódi [Nas16] and with Propositions 2.4.1 and 2.4.4 at hand, we can now give a proof of Theorem 2.4.6.

Proof of Theorem 2.4.6. Let $K \subset \mathbb{R}^n$ be an isotropic convex body and $X_1, \dots, X_n$ be independent random vectors uniformly distributed on K. Denote by T the simplex $\operatorname{conv}\{0, X_1, \dots, X_n\}$ and by u its barycenter; i.e., $u = \frac{1}{n+1}\sum_{i=1}^n X_i$. By Proposition 2.4.1 there is an absolute constant $c_1 > 0$ such that

$$\mathbb{P}\left\{\|u\|_2 \leqslant c_1 L_K\right\} > 1 - \frac{1}{2}e^{-n}.$$

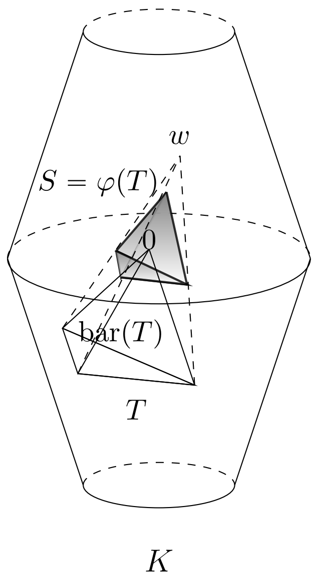

Figure 2.4: Construction involved in the proof of Theorem 2.4.6.

On the other hand, by Proposition 2.4.4, we know that there is an absolute constant $c_2 > 0$ such that

$$\mathbb{P}\left\{|T| \geqslant \frac{c_2^n L_K^n}{n^{\frac{n}{2}}}\right\} > 1 - \frac{1}{2}e^{-n}.$$

By the result of Kannan, Lovász and Simonovits, Lemma 1.3.4, we have that

$$\sqrt{\frac{n+2}{n}} L_K B_2^n \subset K$$

Therefore, the vector $w := -\frac{1}{c_1}u$ belongs to K with probability greater than $1 - \frac{1}{2}e^{-n}$.

It is easy to check that if we apply the homothetic transformation with center w and ratio

$$\lambda = \frac{\|w\|}{\|w-u\|} = \frac{\|w\|}{\|w\| + \|u\|} = \frac{1}{1+c_1} > 0$$

to the simplex T, we obtain another simplex S with barycenter at the origin (see the Figure 2.4) such that

$$|S| \geqslant \lambda^n |T| \geqslant \lambda^n \cdot \frac{c_2^n L_K^n}{n^{\frac{n}{2}}}.$$

Denote by $\bar{X} := \frac{1}{n+1}\sum_{i=1}^{n} X_i$. Therefore, the function $f_n : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n} \to \mathcal{S}_0^n$ we are looking for can be defined by

$$\begin{aligned} f_n(X_1, \ldots, X_n) &:= \varphi(\operatorname{conv}\{0, X_1, \ldots, X_n\}) \\ &= \frac{1}{1+c_1}\operatorname{conv}\{-\bar{X}, X_1 - \bar{X}, \ldots, X_n - \bar{X}\}. \end{aligned}$$

This concludes the proof. □

We can deduce two corollaries from Theorem 2.4.6 that are slight variants of Theorems 2.3.5 and 2.3.4. The first one is a direct aplication of the theorem.

Corollary 2.4.7. *Let $K \subset \mathbb{R}^n$, then*

$$S_\circ^{inn}(K) \preceq \frac{\sqrt{n}}{L_K}. \tag{2.25}$$

The proof of the second one requires the same use of polarity as in the deduction of Theorem 2.3.4 from 2.3.5. That is the reason of the presence of $L_{D(K)^\circ}$ instead of L_{K°.

Corollary 2.4.8. *Let $K \subset \mathbb{R}^n$, then*

$$S_\circ^{out}(K) \leq c\frac{\sqrt{n}}{L_{D(K)^\circ}}. \tag{2.26}$$

2.5 The case of the cube and Dvoretzky-Rogers parallelepiped

As mentioned, for $n = 2$ the cube has the largest volume ratio (with respect to the simplex of minimal volume containing it); for $n = 3$ the same is conjectured. One should expect that a similar phenomenon occurs in high dimensions but, as we have seen in Example 2.1.2, the volume ratio of the cube is uniformly bounded. Moreover, we will show that the simplex can be taken with the same barycenter as the cube.

Proposition 2.5.1. *Let $K := [-\frac{1}{2}, \frac{1}{2}]^n$ the volume one centered cube. Then $S_\circ^{out}(K) \sim 1$.*

Proof. To construct the adequate simplex we need to stretch a little bit the one involved in Example 2.1.2 in order to center it. Denote by $\mathbb{1}$ the vector in $\mathbb{R}^n$ defined as $\sum_{j=1}^{n} e_j$. Consider the simplex

$$S := \operatorname{conv}\{-\frac{n}{2}\mathbb{1}, ne_1 - \frac{1}{2}\mathbb{1}, ne_2 - \frac{1}{2}\mathbb{1}, \ldots, ne_n - \frac{1}{2}\mathbb{1}\}.$$

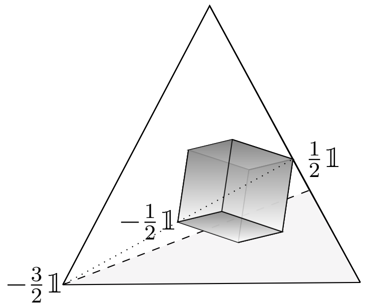

Figure 2.5: The simplex $\operatorname{conv}\{-\frac{3}{2}\mathbb{1}, 3e_1 - \frac{1}{2}\mathbb{1}, 3e_2 - \frac{1}{2}\mathbb{1}, 3e_3 - \frac{1}{2}\mathbb{1}\}$ enclosing the cube $[-\frac{1}{2}, \frac{1}{2}]^3 \subset \mathbb{R}^3$ as in Proposition 2.5.1.

Since K is centrally symmetric, $\operatorname{bar}(K) = 0$. Computing the mean of the vertices (Lemma 2.3.6) we see that the same holds for $\operatorname{bar}(S)$. Observe also that the cube $[-\frac{1}{2}, \frac{1}{2}]^n$ is included in the simplex $T := \operatorname{conv}\{-\frac{1}{2}\mathbb{1}, ne_1 - \frac{1}{2}\mathbb{1}, ne_2 - \frac{1}{2}\mathbb{1}, \dots, ne_n - \frac{1}{2}\mathbb{1}\}$. Indeed, all the points that lie in the cube have coordinates greater than or equal to $-\frac{1}{2}$ and their sum is less than or equal to $\frac{n}{2}$. It remains to see that the point $-\frac{1}{2}\mathbb{1}$ belongs to S, but $-\frac{1}{2}\mathbb{1}$ is exactly $t(-\frac{n}{2})\mathbb{1} + (1-t)\frac{1}{2}\mathbb{1}$ for $t = \frac{2}{n+1}$. An easy computation of a determinant and Stirling formula proves that the volume of S is exactly $\frac{n^n(n+1)}{2n!} \sim 1$. □

We are now going to prove in a non-probabilistic way Theorem 2.3.4. The argument goes as follows: first use Theorem 2.3.2 (see also [PS91]) in the same way that in [DR50] to prove that every convex body K can be enclosed by a parallelepiped of adequate volume and then we make use of the simplex ratio for the cube to conclude the desired bound.

Proof of Theorem 2.3.5. Again, applying the Rogers-Shephard inequality, Theorem 1.2.3, we can suppose without loss of generality that K is centrally symmetric. Suppose also that K is in John's position and, as in the proof of Proposition 2.3.3, take the contacts points $\{v_1, \dots, v_n\}$ given by Theorem 2.3.2. Let P be the parallelepiped

$$P := \bigcap_{i=1}^{n} \{x \in \mathbb{R}^n \ : \ |\langle x, v_i \rangle| \leqslant 1\}.$$

Note that for all $1 \leqslant i \leqslant n$, $\{x \in \mathbb{R}^n \ : \ |\langle x, v_i \rangle| = 1\}$ is a support hyperplane of K and hence, $P \supset K$. The volume of P is given by $|P| = \frac{2^n}{det|v_1,\dots,v_n|}$. Computing the determinant as in equations (2.22) and (2.11) we have that $|P|^{\frac{1}{n}} \sim 1$. Using the fact that $B_2^n \subset K$ we get

$$\left(\frac{|P|}{|K|}\right)^{\frac{1}{n}} \leqslant \left(\frac{|P|}{|B_2^n|}\right)^{\frac{1}{n}} \preceq \sqrt{n}. \tag{2.27}$$

The result now follows combining equation (2.27) and the bound given in Proposition 2.5.1 for the simplex containing the parallelepiped P (with, of course, the fact that the volume ratio is an affine invariant). □

Comparing the result obtained with this technique with (2.25) and (2.26), one should note that the isotropic constant is missing (maybe in case the isotropic constant conjecture is false, (2.25) or (2.25) could give better estimates for certain bodies).

2.5.1 Random Dvoretzy-Rogers Parallelepiped

Observe that, in general, understanding how the parallelepiped P in equation (2.27) looks like seems difficult (its construction depends on certain contact points when L is in John's position, which are not easy to find explicitly), thus Theorem 2.4.6 seems much stronger since it provides a random algorithm that works with high probability.

We therefore state the following novel probabilistic construction of the Dvoretzky-Rogers' parallelepiped, which can be derived from a result of Pivovarov.

Theorem 2.5.2. *Let $L \subset \mathbb{R}^n$ be a centrally symmetric convex body such that L° is in isotropic position and consider the random matrix $T := \sum_{j=1}^n X_j \otimes e_j$, where $X_1, \ldots, X_n$ are independently chosen accordingly to the uniform measure in the isotropic body L°. With probability greater than or equal to $1 - e^{-n}$, the parallelepiped $P = T^{-1}(B_\infty^n)$ contains L and*

$$\left(\frac{|P|}{|L|}\right)^{\frac{1}{n}} \preceq \frac{\sqrt{n}}{L_{L^\circ}}.$$

Proof. First observe that $|co\{0, X_1, \ldots, X_n\}| = \frac{det\left(\sum_{i=1}^n X_i \otimes e_i\right)}{n!}$ and hence Lemma 2.4.5 asserts that

$$\mathbb{P}\left\{ |\det\big(\sum_{j=1}^n X_j \otimes e_j\big)|^{1/n} \geqslant c\sqrt{n} L_{L^\circ} \right\} > 1 - e^{-n}, \tag{2.28}$$

for some absolute constant $c > 0$.

On the other hand since $|\langle X_i, y\rangle| \leqslant 1$ for all $y \in L$ and $1 \leqslant i \leqslant n$ we have that $\|T : X_L \to \ell_\infty^n\| \leqslant 1$, where $T := \sum_{j=1}^n X_j \otimes e_j$.

Thus, $T(L) \subset B_\infty^n$, or equivalently $L \subset T^{-1}(B_\infty^n) := P$ and the ratio

$$\left(\frac{|P|}{|L|}\right)^{\frac{1}{n}} = \frac{|B_\infty^n|^{\frac{1}{n}}}{|\det T|^{\frac{1}{n}} |L|^{\frac{1}{n}}}. \tag{2.29}$$

Therefore, by equations (2.29) and (2.28) and taking into account that $|L|^{\frac{1}{n}} \sim \frac{1}{n}$ (which comes from an application of the Blaschke-Santaló/Bourgain-Milman inequality, since $|L^\circ| = 1$) we have, with probability greater than or equal to $1 - e^{-n}$,

$$\left(\frac{|P|}{|L|}\right)^{\frac{1}{n}} \leqslant c\frac{\sqrt{n}}{L_{L^\circ}}, \tag{2.30}$$

which concludes the proof. $\square$

Chapter 3

General bounds

In this chapter we discuss the problem of approximating a convex body by another one in a more general setting. We define the volume ratio of a pair of convex bodies, that measures how well can one of them be approximated by an affine image of the other. In Section 3.1 we present the basic definitions and elementary properties of this quantity. We also define the largest volume ratio of a convex body and show the best known bound so far. In Section 3.2 we present a series of examples for which this bound can be improved. In Section 3.3 we introduce some stochastic tools that allow us to extend our result to some natural classes of convex bodies such as unit balls of unitary invariant norms and tensor products of ℓ_p-spaces.

3.1 General volume ratio

We will discuss a natural generalization of the problems treated in the previous chapter: replacing the simplex by another affine class of convex bodies. For $K, L \subset \mathbb{R}^n$ we ask whether L can be enclosed by an affine image of K of similar volume. This kind of approximation was studied by MacBeath [Mac51a] in the context of modified Banach-Mazur distances. We will work with the following definition introduced by Giannopoulos and Hartzoulaki [GH02] and also studied by Gordon, Litvak, Meyer and Pajor [GLMP04]: given two convex bodies K and L in $\mathbb{R}^n$ the *volume ratio* of the pair (K, L) is defined as

$$\operatorname{vr}(K, L) := \inf \left\{ \left(\frac{|K|}{|T(L)|} \right)^{\frac{1}{n}} : T(L) \text{ is contained in } K \right\}, \tag{3.1}$$

where the infimum (actually a minimum) is taken over all affine transformations T. In other words, $\operatorname{vr}(K, L)$ measures how well can K be approximated by an affine image of L. Note that the classic value $\operatorname{vr}(K)$ is just $\operatorname{vr}(K, B_2^n)$ where B_2^n is the Euclidean unit ball in $\mathbb{R}^n$.

Given a convex body K, it is natural to ask how "good" an approximation of this kind can be (in terms of the dimension of the ambient space). Namely, we want to known *how large the value* $\mathrm{vr}(K, L)$ *is* (for arbitrary convex bodies $L \subset \mathbb{R}^n$). Thus, it is important to compute the *largest volume ratio of* K, given by

$$\mathrm{lvr}(K) := \sup_{L \subset \mathbb{R}^n} \mathrm{vr}(K, L),$$

where the sup runs over all the convex bodies L. The best general bound known so far for this quantity is due to Giannopoulos and Hartzoulaki [GH02]. They proved that for every convex body $K \subset \mathbb{R}^n$

$$\mathrm{lvr}(K) \preceq \log(n)\sqrt{n}. \tag{3.2}$$

We will return to their result in Section 3.3. We first observe that for many bodies $K \subset \mathbb{R}^n$, the logarithmic factor in (3.2) can be removed having that

$$\mathrm{lvr}(K) \preceq \sqrt{n}. \tag{3.3}$$

In fact, there are not examples of convex bodies for which the largest volume ratio is asymptotically strictly larger that the square root of the dimension of the ambient space. We will show many examples of natural classes of convex bodies for which we can achieve the bound (3.3). In all cases this bound is sharp, since, as we will see in Chapter 4, we always have that

$$\mathrm{lvr}(K) \succeq \sqrt{n}.$$

3.1.1 Elementary properties

We now recall some elementary properties of the volume ratio that follow directly from the definition. First notice that, as in the case of the simplex, the volume ratio is invariant under affine transformations. In other words, the volume ratio between K and L depends exclusively on the affine classes of the bodies involved. Namely, given two affine transformations $A, B \in GL(n, \mathbb{R})$,

$$\begin{aligned}
\mathrm{vr}(AK, BL) = \min_{TBL \subset AK} \left(\frac{|AK|}{|TBL|}\right)^{1/n} &= \min_{TBL \subset K} \left(\frac{\det(A)|K|}{|TBL|}\right)^{1/n} \\
&= \min_{TBL \subset K} \left(\frac{|K|}{\det(A)^{-1}|TBL|}\right)^{1/n} \\
&= \min_{TBL \subset K} \left(\frac{|K|}{|A^{-1}TBL|}\right)^{1/n} \\
&= \min_{\tilde{T}L \subset K} \left(\frac{|K|}{|\tilde{T}L|}\right)^{1/n} = \mathrm{vr}(K, L).
\end{aligned}$$

Another useful property of the volume ratio is a sort of multiplicative triangle inequality.

Proposition 3.1.1. *Given $K, L \subset \mathbb{R}^n$, we have*

$$\mathrm{vr}(K,L) \leqslant \mathrm{vr}(K,Z) \cdot \mathrm{vr}(Z,L)$$

for every convex body Z in $\mathbb{R}^n$.

Proof. Suppose that $TZ \subset K$ and $SL \subset Z$, then we have that $TSL \subset K$ and,

$$\mathrm{vr}(K,L) \leqslant \left(\frac{|K|}{|TSL|}\right)^{\frac{1}{n}} = \left(\frac{|K|}{|TZ|}\right)^{\frac{1}{n}} \left(\frac{|Z|}{|SL|}\right)^{\frac{1}{n}}.$$

Taking infimum on the right side we get,

$$\mathrm{vr}(K,L) \leqslant \mathrm{vr}(K,Z)\,\mathrm{vr}(Z,L),$$

which completes the proof. □

In the case that K and L are centrally symmetric we can relate their volume ratio with the norm of operators between the spaces X_K and X_L. We can also apply Blaschke-Santaló and Bourgain-Milman (Theorems 1.2.1 and 1.2.2) inequalities to relate it to the volume ratio between their polar bodies.

Proposition 3.1.2. *For every pair of centrally symmetric convex bodies (K,L) in $\mathbb{R}^n$ the following holds:*

1. $$\mathrm{vr}(K,L) = \left(\frac{|K|}{|L|}\right)^{\frac{1}{n}} \cdot \inf_{T \in SL(n,\mathbb{R})} \|T : X_L \to X_K\|,$$

   where the infimum runs over all the linear transformations T that lie on the special linear group of degree n (matrices of determinant one).

2. *If $T : X_L \to X_K$ is a linear operator we have that $\frac{1}{\|T:X_L\to X_K\|} \cdot T(L) \subset K$ and so*

   $$\mathrm{vr}(K,L) \leqslant \frac{\|T : X_L \to X_K\| |K|^{\frac{1}{n}}}{|\det T|^{\frac{1}{n}} |L|^{\frac{1}{n}}}.$$

3. $\mathrm{vr}(K,L) \sim \mathrm{vr}(L^\circ, K^\circ)$.

Proof. For (1) first notice that, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is an affine transformation, say $f = T + v$ with T linear and $v \in \mathbb{R}^n$, one has that if $f(L) \subset K$ then $T(L) \subset K$. In fact, since L is centrally symmetric, $f(L)$ is symmetric with respect to v. So, given $x \in L$, both $Tx + v$ and $v - T(x)$ lie in $f(L) \subset K$. Since K is itself centrally symmetric, we have that $T(x) - v \in K$ and so, by convexity, $T(x) \in K$. As $|f(L)| = |TL|$ we can compute the volume ratio considering only linear operators. In that case, the condition that $T(L) \subset K$ can be written as $\|T : X_L \to X_K\| \leqslant 1$, where $\|T : X_L \to X_K\|$ denotes the operator norm of T between the spaces X_L and X_K.

$$
\begin{aligned}
\mathrm{vr}(K, L) &= \inf_{\|T:X_L\to X_K\|\leqslant 1} \left(\frac{|K|}{|TL|}\right)^{\frac{1}{n}} = \inf_{T\in GL(n,\mathbb{R})} \left(\frac{|K|}{|\frac{T}{\|T:X_L\to X_K\|}L|}\right)^{\frac{1}{n}} \\
&= \inf_{T\in GL(n,\mathbb{R})} \left(\frac{|K|}{|L|}\right)^{\frac{1}{n}} \frac{\|T : X_L \to X_K\|}{(\det T)^{\frac{1}{n}}} \\
&= \inf_{T\in GL(n,\mathbb{R})} \left(\frac{|K|}{|L|}\right)^{\frac{1}{n}} \left\|\frac{T}{(\det T)^{\frac{1}{n}}} : X_L \to X_K\right\| \\
&= \left(\frac{|K|}{|L|}\right)^{\frac{1}{n}} \inf_{T\in SL(n,\mathbb{R})} \|T : X_L \to X_K\|.
\end{aligned}
$$

Observe that (2) follows directly from the definitions of the operator norm.

Property (3) is a direct consequence of Blaschke-Santaló and Bourgain-Milman inequalities (Theorems 1.2.1 and 1.2.2) and the fact that if $TL \subset K$, $T^*K^\circ \subset L^\circ$. Indeed,

$$
\begin{aligned}
\mathrm{vr}(K, L) &= \inf_{TL\subset K} \left(\frac{|K|}{|TL|}\right)^{\frac{1}{n}} \left(\frac{|K^\circ|}{|K^\circ|}\right)^{\frac{1}{n}} \left(\frac{|L^\circ|}{|L^\circ|}\right)^{\frac{1}{n}} \\
&= \inf_{TL\subset K} \left(\frac{|L^\circ|}{|T^*K^\circ|}\right)^{\frac{1}{n}} \left(\frac{|K||K^\circ|}{|L||L^\circ|}\right)^{\frac{1}{n}} \sim \mathrm{vr}(L^\circ, K^\circ).
\end{aligned}
$$

□

Notice that by Rogers-Shephards inequality Theorem 1.2.3, for every convex body $L \subset \mathbb{R}^n$ we have $\mathrm{vr}(L - L, L) \leqslant 4$. Therefore, by Proposition 3.1.1

$$\mathrm{vr}(K, L) \leqslant \mathrm{vr}(K, L - L) \cdot 4. \tag{3.4}$$

Thus, the largest volume ratio of the body K can be estimated by considering the sup over all symmetric bodies. Precisely,

$$\mathrm{lvr}(K) \leqslant 4 \sup_{L\subset\mathbb{R}^n} \mathrm{vr}(K, L), \tag{3.5}$$

where the sup runs over all the *centrally symmetric convex bodies* L. This will be useful since it allow us to deal only with bodies which are centrally symmetric.

3.2 Examples

In order to obtain upper bounds for the largest volume ratio for some natural classes of convex bodies we need to introduce some tools. Before doing that we review some examples where we can easily bound it. Recall that if a centrally symmetric convex body $K \subset \mathbb{R}^n$ is in John's position we have that

$$B_2^n \subset K \subset \sqrt{n} B_2^n$$

and hence $vr(B_2^n, K) \leq \sqrt{n}$. Example 2.3.1 shows that $\mathrm{vr}(B_2^n, S) \sim \sqrt{n}$ for a simplex S. Then,

$$\mathrm{lvr}(B_2^n) \sim \sqrt{n}.$$

In the previous chapter we proved that for a simplex $S \subset \mathbb{R}^n$ we have $\mathrm{vr}(S, K) \leq \sqrt{n}$ for every convex body $K \subset \mathbb{R}^n$. Since, by Example 2.1.1, $\mathrm{vr}(S, B_2^n) \sim \sqrt{n}$ we conclude that

$$\mathrm{lvr}(S) \sim \sqrt{n}.$$

Recall that given K, the Dvoretzky and Rogers parallelepiped (see equation (2.27)) fullfils that $K \subset P$ and $\mathrm{vr}(P, K) \leqslant \sqrt{n}$. Since every parallelepiped is an affine image of the cube B_∞^n, we have $\mathrm{lvr}(B_\infty^n) \leq \sqrt{n}$. As $\mathrm{vr}(B_\infty^n, B_2^n) \sim \sqrt{n}$ we get that

$$\mathrm{lvr}(B_\infty^n) \sim \sqrt{n}.$$

3.2.1 Polytopes

The next proposition was obtained by Bárány and Füredi [BF88], Carl and Pajor [CP88] and Gluskin [Glu89] applying different techniques (all in 1988), it bounds the volume of a polytope contained in the Euclidean ball.

Lemma 3.2.1. *Let $v_1, \ldots, v_N \in B_2^n$, then, for $P := \mathrm{conv}\{v_1, \ldots, v_N\}$, we have,*

$$|P|^{\frac{1}{n}} \preceq \frac{\sqrt{\log(1 + \frac{N}{n})}}{n}. \tag{3.6}$$

To prove the lemma we are going to use a result of Sidák regarding the Gaussian measure of intersections of *symmetric strips*, which are sets of the form

$$P = \{x \in \mathbb{R}^n \mid |\langle x, v\rangle| \leqslant \alpha\},$$

for $\alpha > 0$ and $v \in \mathbb{R}^n$. A proof of it can be found in [AAGM15, Theorem 4.4.5].

Proposition 3.2.2 (Sidák). *If $P_1, \dots, P_N$ are symmetric strips in $\mathbb{R}^n$ then*

$$\gamma_n\left(\bigcap_{i=1}^N P_i\right) \geqslant \prod_{i=1}^N \gamma_n(P_i),$$

where γ_n is the Gaussian measure in $\mathbb{R}^n$.

Proof of Lemma 3.2.1. We will prove that if

$$K = \{x \in \mathbb{R}^n \mid |\langle x, v_i \rangle| \leqslant 1, \text{for all } 1 \leqslant i \leqslant N\},$$

then

$$|K|^{\frac{1}{n}} \succeq \frac{1}{\sqrt{\log(1+\frac{N}{n})}}. \tag{3.7}$$

The result then follows by polarity and applying Balschke-Santaló inequality (Theorem 1.2.1). Indeed, $P \subset K^\circ$ and hence

$$|P|^{\frac{1}{n}} \leqslant |K^\circ|^{\frac{1}{n}} \preceq \frac{1}{n|K|^{\frac{1}{n}}}.$$

In order to prove (3.7) set $\alpha > 0$ to be choosen later and consider the symmetric strips

$$P_i = \{x \in \mathbb{R}^n \mid |\langle x, v_i \rangle| \leqslant \alpha\}.$$

The width of each P_i is given by $\frac{2\alpha}{\|v_i\|_2} \geqslant 2\alpha$. So, if γ_n is the Gaussian measure on $\mathbb{R}^n$,

$$\gamma_n(P_i) \geqslant \frac{1}{\sqrt{2\pi}} \int_{-\alpha}^{\alpha} e^{\frac{-t^2}{2}}\, dt \geqslant 1 - e^{\frac{-\alpha^2}{2}}.$$

Now, notice that $\alpha K = \bigcap_{i=1}^N P_i$ and so

$$\gamma_n(\alpha K) = \gamma_n\left(\bigcap_{i=1}^N P_i\right) \geqslant \left(1 - e^{\frac{-\alpha^2}{2}}\right)^N.$$

Since $|\alpha K| \geqslant (2\pi)^{\frac{n}{2}} \gamma_n(\alpha K)$, if we choose $\alpha = 2\sqrt{\log(1+\frac{N}{n})}$ and take the nth-root we have

$$\begin{aligned} 2\sqrt{\log(1+\frac{N}{n})}|K|^{\frac{1}{n}} &\geqslant \sqrt{2\pi}\left(1 - e^{-2\log(1+\frac{N}{n})}\right)^{N/n} \\ &= \sqrt{2\pi}\left(1 - \frac{1}{(1+\frac{N}{n})^2}\right)^{N/n} \geqslant c \end{aligned}$$

which completes the proof. $\square$

Notice that Sidák's result is actually a consequence of the recently proved Gaussian inequality [Roy14] (see also[LM17]), which asserts that for any pair of centrally symmetric convex bodies C_1 and C_2 one has $\gamma_n(C_1 \cap C_2) \geqslant \gamma_n(C_1)\gamma_n(C_2)$.

We can use the previous lemma to bound the largest volume ratio for polytopes with few vertices.

Proposition 3.2.3. *Let $c > 0$ and $v_1, \ldots, v_N \in \mathbb{R}^n$ with $N = \lceil cn \rceil$. If $P := \operatorname{conv}\{v_1, \ldots, v_N\}$ then $\operatorname{lvr}(P) \preceq \sqrt{n}$.*

Proof. Assume P in Löwner's position and consider the simplex S whose vertices are the contact points given by Theorem 2.3.2. We have that $S \subset P$. Using equation (2.11) we have that $|S|^{\frac{1}{n}} \preceq \frac{1}{n}$ and hence, applying Lemma 3.2.1,

$$\left(\frac{|P|}{|S|}\right)^{\frac{1}{n}} \preceq \sqrt{\log(1 + \frac{n}{cn})} \sim 1. \tag{3.8}$$

Now, given a convex body $K \subset \mathbb{R}^n$, applying Theorem 2.3.4 we know that $\operatorname{vr}(S, K) \preceq \sqrt{n}$. So, by Proposition 3.1.1,

$$\operatorname{vr}(P, K) \leqslant \operatorname{vr}(P, S)\operatorname{vr}(S, K) \preceq \sqrt{n}.$$

□

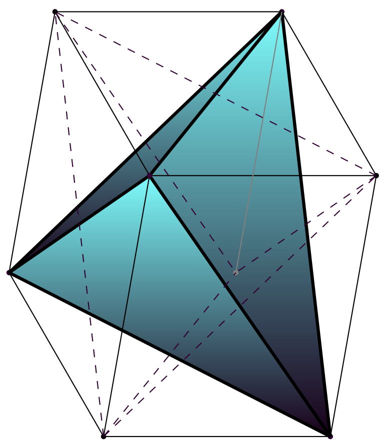

Figure 3.1: Simplex formed by certain vertices of a polytope as in the proof of Proposition 3.2.3.

3.2.2 Unconditional bodies

We say that a convex body K is *unconditional* if $(x_1, \ldots, x_n) \in K$ implies $(\varepsilon_i x_1, \ldots, \varepsilon_n x_n) \in K$ for any choice of sings $\varepsilon_i \in \{-1, 1\}$. Notice that an unconditional convex body is necessarily centrally symmetric and $\|(x_1, \ldots, x_n)\|_K = \|(\varepsilon_i x_1, \ldots, \varepsilon_n x_n)\|_K$. We also say that $\| \cdot \|_K$ is an *unconditional norm* .

We say that a body is *permutationally symmetric* if $(x_1, \ldots, x_n) \in K$ implies $(\varepsilon_i x_{\sigma(1)}, \ldots, \varepsilon_n x_{\sigma(n)}) \in K$ for any choice of sings $\varepsilon_i \in \{-1, 1\}$ and any permutation σ we also say that $\| \cdot \|_K$ is a *permutationally symmetric norm* (is more common to use the term symmetric norm, but we want to avoid any confusion with centrally symmetric bodies). The following lemma, that is a consequence of a more general result due to Lozanovskii [Loz69] (see also [TJ89, Lemma 39.3]), allows us to bound the volume ratio for unconditional convex bodies.

Lemma 3.2.4. *Let $K \subset \mathbb{R}^n$ be centrally symmetric convex body, there are positive numbers $a_1, \ldots, a_n$ such that*

$$\|(a_1, \ldots, a_n)\|_K = 1 \quad and \quad \|(a_1^{-1}, \ldots, a_n^{-1})\|_{K^\circ} = n. \tag{3.9}$$

Moreover, if K is permutationnaly symmetric, $a_1 = \cdots = a_n = \frac{1}{\|(1,\ldots,1)\|_K}$.

Proof. Consider the function $\Phi(t_1, \ldots, t_n) := (\prod_{i=1}^n t_i)^{\frac{1}{n}}$ on the positive orthant. Since it is strictly concave and 1-homogeneous it has a unique maximum over ∂K. Set,

$$\Phi_0 = \Phi(a_1, \ldots, a_n) = \max\{\Phi(t_1, \ldots, t_n) | \|(t_1, \ldots, t_n)\|_K = 1\},$$

and $A = \{(t_1 \ldots, t_n) | \Phi((t_1 \ldots, t_n) \geqslant \Phi_0\}$. Since A is strictly convex and K is convex, there is $(v_1, \ldots, v_n)$ that separates both sets, i.e. $\|(v_1, \ldots, v_n)\|_{K^\circ} = 1$ and $\langle (v_1, \ldots, v_n), (t_1, \ldots, t_n) \rangle > 1$ for $(t_1, \ldots, t_n) \in A$, $(t_1, \ldots, t_n) \neq (a_1, \ldots, a_n)$. Since Φ is differentiable and 1-homogeneous, by Euler formula we have that,

$$\langle \nabla\Phi(a_1, \ldots, a_n), (a_1, \ldots, a_n) \rangle = \Phi_0.$$

The last equality implies that

$$\langle \nabla\Phi(a_1, \ldots, a_n), (t_1, \ldots, t_n) \rangle < \Phi_0$$

for all $(t_1, \ldots, t_n) \in K$, and then,

$$(v_1, \ldots, v_n) = \frac{\nabla\Phi(a_1, \ldots, a_n)}{\Phi} = \frac{1}{n}(a_1^{-1}, \ldots, a_n^{-1}).$$

Hence, $\|(a_1^{-1}, \ldots, a_n^{-1})\|_{K^\circ} = n$.

If K is permutationally symmetric, since Φ is invariant under permutations, by the uniqueness of the maximum, we must have $a_1 = \cdots = a_n$, what completes the proof. $\square$

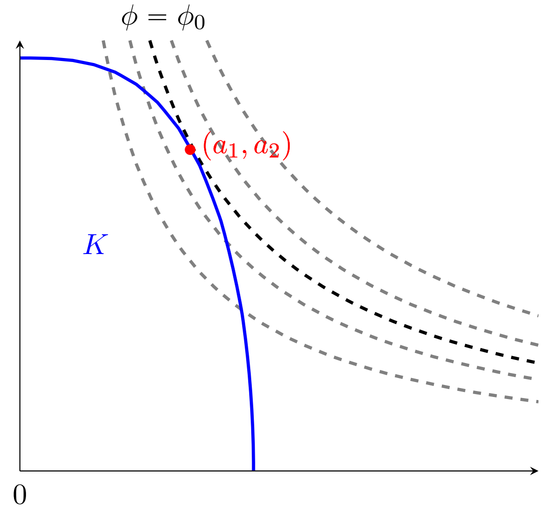

Figure 3.2: Level curves of ϕ as in the proof of Lemma 3.2.4

Remark 3.2.5. We can interpret the last result geometrically. Let $K \subset \mathbb{R}^n$ be an unconditional convex body, consider the numbers $a_1, \dots, a_n$ given by Lemma 3.2.4 and define D_K as the diagonal operator with entries $\{a_i\}_{1 \leqslant i \leqslant n}$, then equation (3.9) together with unconditionality mean that $D_K B_\infty^n \subset K$ and $D_K^{-1} B_\infty^n \subset nK^\circ$, which implies,

$$D_K B_\infty^n \subset K \subset n D_K B_1^n. \tag{3.10}$$

Taking volumes in equation (3.10), we have that,

$$\det(D_K)|B_\infty^n|^{\frac{1}{n}} \leqslant |K|^{\frac{1}{n}} \leqslant \det(D_K) n |B_1^n|^{\frac{1}{n}}.$$

Since, $|B_\infty^n|^{\frac{1}{n}} \sim n|B_1^n|^{\frac{1}{n}}$, we conclude that $\mathrm{vr}(K, B_\infty^n) \sim 1$.

If we combine this fact with Theorem 2.5.2 we obtain the following result.

Corollary 3.2.6. *Given an unconditional convex body K, set D_K the diagonal operator as in Remark 3.2.5. Let $L \subset \mathbb{R}^n$ be a centrally symmetric convex body such that L° is in isotropic position and consider the random matrix*

$$T := \sum_{j=1}^{n} X_j \otimes e_j,$$

where $X_1, \dots, X_n$ are independently chosen accordingly to the uniform measure in the isotropic body L°. With probability greater than or equal to $1 - e^{-n}$, for every unconditional isotropic body $K \subset \mathbb{R}^n$, the position $\tilde{L} :=$

$D_K T(L)$ lies inside K and

$$\left(\frac{|K|}{|\tilde{L}|}\right)^{\frac{1}{n}} \leq \frac{\sqrt{n}}{L_{L^\circ}}. \tag{3.11}$$

Proof. By Theorem 2.5.2 we know that with probability greater than $1-e^{-n}$, $T(L) \subset (B_\infty^n)$ and

$$\left(\frac{|P|}{|L|}\right)^{\frac{1}{n}} \leq \frac{\sqrt{n}}{L_{L^\circ}}.$$

Hence $D_K T(L) \subset D_K(B_\infty^n) \subset K$ and the result follows from the fact that $\left(\frac{|K|}{|D_K(B_\infty^n)|}\right)^{\frac{1}{n}} \sim 1$. □

It is worth to observe that if K is an isotropic unconditional convex body we can ensure the existence of a "large" cube inside K. The next proposition is due to Bobkov and Nazarov [BN03].

Proposition 3.2.7. *Let $K \subset \mathbb{R}^n$ be an unconditional isotropic convex body. Then, $[-\frac{L_K}{\sqrt{2}}, \frac{L_K}{\sqrt{2}}]^n \subset K$.*

Proof. Set $K^+ = 2K \cap \mathbb{R}^n_+$. The barycenter $v = (v_1, \ldots, v_n)$ of K^+ lies in K^+, then the rectangle $[0, v_1] \times \cdots \times [0, v_n] \subset K^+$. Since K is unconditional, applying Khintchine inequality, we have that

$$4L_K^2 = \int_{K^+} x_i^2 dx \leqslant 2\left(\int_{K^+} x_i dx\right)^2.$$

So, if we compute the coordinates of v we get

$$v_i = \int_{K^+} x_i dx \geqslant \sqrt{2} L_k,$$

which implies the desired inclusion. □

Notice that, since $\frac{L_K}{\sqrt{2}} \geqslant \frac{1}{2\sqrt{e\pi}}$ (see Proposition 1.3.3), we have

$$\left[-\frac{1}{2\sqrt{e\pi}}, \frac{1}{2\sqrt{e\pi}}\right]^n \subset K.$$

We obtain the following corollary.

Corollary 3.2.8. *Let $L \subset \mathbb{R}^n$ be a centrally symmetric convex body such that L° is in isotropic position and consider the random matrix*

$$T := \sum_{j=1}^n X_j \otimes e_j,$$

where $X_1, \dots, X_n$ are independently chosen accordingly to the uniform measure in the isotropic body L°. With probability greater than or equal to $1 - e^{-n}$, for every unconditional isotropic body $K \subset \mathbb{R}^n$, the position $\tilde{L} := \frac{1}{2\sqrt{\pi e}} \cdot T(L)$ lies inside K and

$$\left(\frac{|K|}{|\tilde{L}|}\right)^{\frac{1}{n}} \preceq \frac{\sqrt{n}}{L_{L^\circ}}. \tag{3.12}$$

Observe that in Corollary 3.2.8 we avoid the diagonal operator D_K adding the condition that K is isotropic. One can find the isotropic position of an unconditional convex body computing $\int_K x_i^2$ for $1 \leqslant i \leqslant n$. Hence, in the cases when one can compute that quantity, Corollary 3.2.8 seems more useful than Corollary 3.2.6.

3.3 Bounds for some natural classes of convex bodies

3.3.1 Rudelson's position

Given a convex body $W \subset \mathbb{R}^n$ we need to introduce a position $\tilde{W}$ highly related with the well-known ℓ-position (see Theorem 1.3.6). It has been introduced by Rudelson in [Rud00] and its existence can be also tracked in the proof of the main theorem of the paper of Giannopoulos and Hartzoulaki [GH02]. They use this position together with Chevet's inequality to bound the volume ratio. They prove that for every convex body $K \subset \mathbb{R}^n$, $\mathrm{lvr}(K) \preceq \log(n)\sqrt{n}$. So far, this is the best known general bound. However is not known to be optimal, and, in fact for a wide class of natural convex bodies (many of which we have already seen) the largest volume ratio can be bounded by the square root of the dimension of the ambient space.

Proposition 3.3.1. *Given a centrally symmetric convex body $W \subset \mathbb{R}^n$ there is a position of W,$\tilde{W}$ that satisfies:*

- $\ell(\tilde{W}) \preceq \sqrt{n}\log(n)$,
- $\ell(\tilde{W}^\circ) \preceq \sqrt{n}$,
- $\|id : \ell_2^n \to X_{\tilde{W}^\circ}\| \preceq \frac{\sqrt{n}}{\log(n)}$.

In particular,

$$\frac{1}{|\tilde{W}|^{\frac{1}{n}}} \leqslant \ell(\tilde{W}) \preceq \sqrt{n}\log(n).$$

When a convex body in $\mathbb{R}^n$ satisfies the previous estimates we say it is in *Rudelson's-position.*

Proof. By Theorem 1.3.6 we can assume that W fulfils

$$\ell(W)\ell(W^\circ) \preceq n\log(n).$$

Since this quantity is invariant under scalar multiplication suppose that $\ell(W) \preceq \sqrt{n}\log(n)$ and $\ell(W^\circ) \preceq \sqrt{n}$.

Choose $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$B_2^n \subset SW^\circ \subset \sqrt{n}B_2^n, \tag{3.13}$$

and consider $T := I + \alpha S$ with $\alpha := \frac{\log(n)}{\sqrt{n}}$. Lets see that $\tilde{W} := (T^*)^{-1}(W)$ is the wanted position. First,

$$\begin{aligned}\ell((T^*)^{-1}(W)) = \ell((TW^\circ)^\circ) = \ell((I+\alpha S)W^\circ)^\circ) &\leqslant \ell(W) + \ell((\alpha SW^\circ)^\circ)\\ &\leqslant \sqrt{n}\log(n) + \alpha\ell((SW^\circ)^\circ).\end{aligned}$$

Observe that by inclusion (3.13) we have that $(SW^\circ)^\circ \supset \frac{1}{\sqrt{n}}B_2^n$ and hence $\ell((SW^\circ)^\circ) \leqslant \sqrt{n}\ell(B_2^n) \preceq n$, what proves the first assertion.

For the second one, observe that since $\ell(\cdot)$ is an operator norm,

$$\ell(\tilde{W}^\circ) = \ell(TW^\circ) = \ell((id+\alpha S)W^\circ) \leqslant \|(id+\alpha S)^{-1} : \ell_2^n \to \ell_2^n\|\ell(W^\circ).$$

Since $id + \alpha S$ is a positive operator, $\|(id+\alpha S)^{-1} : \ell_2^n \to \ell_2^n\| \leqslant 1$.

Finally,

$$\begin{aligned}\|id : \ell_2^n \to X_{\tilde{W}^\circ}\| &= \|(id+\alpha S)^{-1} : \ell_2^n \to X_{W^\circ}\|\\ &\leqslant \|((\alpha S)^{-1} + id)^{-1} : \ell_2^n \to \ell_2^n\|\|((\alpha S)^{-1} : \ell_2^n \to X_{W^\circ}\|\\ &\leqslant \|((\alpha S)^{-1} : \ell_2^n \to X_{W^\circ}\| \leqslant \frac{1}{\alpha}.\end{aligned}$$

The fact that $\frac{1}{|\tilde{W}|^{\frac{1}{n}}} \leqslant \ell(\tilde{W})$ follows directly from Urysohn's inequality, Lemma 1.3.7. □

As seen before there is a relation between the volume ratio of a pair of centrally symmetric convex bodies K and L and the norm of operators from X_L to X_K. The well known Chevet's inequality bounds the expected value of a random Gaussian operator in terms of some geometrical parameters of the bodies K and L. It's states that

Theorem 3.3.2 (Gaussian Chevet's inequality)**.** *Let $A = (g_{ij})_{1\leqslant i,j\leqslant n} \in \mathbb{R}^{n\times n}$ be a random matrix with independent Gaussian entries $g_{ij} \sim \mathcal{N}(0,1)$ and $L, K \subset \mathbb{R}^n$ two convex bodies, then*

$$\mathbb{E}(\|A : X_L \to X_K\|) \preceq (\ell(K)\|id : \ell_2^n \to X_{L^\circ}\| + \|id : \ell_2^n \to X_K\|\ell(L^\circ)). \tag{3.14}$$

To our purposes we will use the following high probability version of the Gaussian Chevet's inequality (tail inequality).

Proposition 3.3.3. *Let $A = (g_{ij})_{1\leqslant i,j\leqslant n} \in \mathbb{R}^{n\times n}$ be a random matrix with independent Gaussian entries $g_{ij} \sim \mathcal{N}(0,1)$ and $K, L \subset \mathbb{R}^n$ two convex bodies. Then, for all $u \geqslant 0$, with probability greater than $1 - e^{-u^2}$ we have*

$$\begin{aligned}\|A : X_L \to X_K\| \leq & \ell(K)\|id : \ell_2^n \to X_{L^\circ}\| + \ell(L^\circ)\|id : \ell_2^n \to X_K\| \\ & + u\|id : \ell_2^n \to X_{L^\circ}\| \cdot \|id : \ell_2^n \to X_K\|.\end{aligned} \tag{3.15}$$

We will give a sketch of the proof of this proposition in Section 3.4. First we will show how Rudelson's position together with Chevet's inequality can be used to bound the largest volume ratio for some natural classes of convex bodies. Observe that, by Proposition 3.1.2 (2), bounding simultaneously the determinant (from below) and the norm (from above) of an operator gives a bound for the volume ratio. We will also need the following lower bound for the determinant of a random Gaussian matrix, which can be found in [Piv10, Corollary 1].

Lemma 3.3.4. *Let $A = (g_{ij})_{1\leqslant i,j\leqslant n} \in \mathbb{R}^{n\times n}$ with $g_{ij} \sim \mathcal{N}(0,1)$, then with probability at least $1 - e^{-n}$ we have*

$$\det(A)^{\frac{1}{n}} \succeq \sqrt{n}. \tag{3.16}$$

Combining the last inequality together with Proposition 3.3.6, we can ensure that for any $u \leqslant \sqrt{n}$, with probability greater than $1 - 2e^{-u^2}$, a random Gaussian operator A fulfils both,

$$\det(A)^{\frac{1}{n}} \succeq \sqrt{n},$$

and

$$\begin{aligned}\|A : X_L \to X_K\| \preceq & \ell(K)\|id : \ell_2^n \to X_{L^\circ}\| + \ell(L^\circ)\|id : \ell_2^n \to X_K\| \\ & + u\|id : \ell_2^n \to X_{L^\circ}\| \cdot \|id : \ell_2^n \to X_K\|,\end{aligned} \tag{3.17}$$

for any pair of convex bodies $K, L \subset \mathbb{R}^n$. Recall also that for $T \in GL(n,\mathbb{R})$, by Proposition 3.1.2 (2),

$$\mathrm{vr}(K,L) \leqslant \frac{\|T : X_L \to X_K\| |K|^{\frac{1}{n}}}{|\det T|^{\frac{1}{n}} |L|^{\frac{1}{n}}}. \tag{3.18}$$

Assuming that L is in Rudelson's position, and combining equations (3.17) and (3.18), we have that $\frac{TL}{\|T\|} \subset K$ and

$$\left(\frac{|K|}{|\frac{TL}{\|T\|}|}\right)^{\frac{1}{n}} \preceq \ell(K)|K|^{\frac{1}{n}}\sqrt{n} + (\log(n) + u)\sqrt{n}\|id : \ell_2^n \to X_K\||K|^{\frac{1}{n}} \tag{3.19}$$

with probability greater than or equal to $1 - 2e^{-u^2}$. Hence, we have the following proposition.

Proposition 3.3.5. *Let $K \subset \mathbb{R}^n$ be a convex body and $0 \leqslant u \leqslant \sqrt{n}$, then*

$$\begin{aligned}\mathrm{lvr}(K) \le \ell(K)|K|^{\frac{1}{n}}\sqrt{n} + \log(n)\sqrt{n}\|id : \ell_2^n \to X_K\| \\ +u\sqrt{n}\|id : \ell_2^n \to X_K\||K|^{\frac{1}{n}}.\end{aligned}$$

In the following we are going to use the last proposition to bound the volume ratio for some classes of convex bodies.

3.3.2 Unitary invariant norms

We will first focus on the unit balls of the Schatten classes. For every matrix $T \subset \mathbb{R}^{d\times d}$ consider $s(T) = (s_1(T), \ldots, s_d(T))$ the sequence of eigenvalues of $(TT^*)^{\frac{1}{2}}$ (the singular values of T). The *p-Schatten norm* of $T \in \mathbb{R}^{d\times d}$ is defined as

$$\sigma_p(T) = \|s(T)\|_{\ell_p^d}; \tag{3.20}$$

that is, the ℓ_p-norm of the singular values of T. The p-Schatten norm arises as a generalization of the classical Hilbert-Schmidt norm. Many different properties of them in the finite dimensional setting have been largely studied in the area of asymptotic geometric analysis. For example, Köning, Meyer and Pajor [KMP98] established the boundedness of the isotropic constants of the unit balls of $\mathcal{S}_p^d \subset \mathbb{R}^{d\times d}$ $(1 \leqslant p \leqslant \infty)$, Guédon and Paouris [GP07] also studied concentration mass properties for the unit balls, Barthe and Cordero-Eurasquin [BCE13] analyzed variance estimates, Radke and Vritsiou [RV16] proved the thin-shell conjecture, and recently Kabluchko, Prochno and Thäle [KPT18] exhibited the exact asymptotic behaviour of the volume and standard volume ratio; just to mention a few.

Therefore it is natural to try to understand what happens with the largest volume ratio of their unit ball.

We write $\mathcal{S}_p^d := (\mathbb{R}^{d\times d}, \sigma_p)$ and denote by $B_{\mathcal{S}_p^d} \subset \mathbb{R}^{d\times d}$ the unit ball of $\mathcal{S}_p^d$. Schatten norms are particular cases of a more general class of norms, *unitary invariant norms.* A unitary invariant norm $\mathcal{N}$ on $\mathbb{R}^{d\times d}$, is a norm that satisfies $\mathcal{N}(UTV) = \mathcal{N}(T)$ for all $U, V \in \mathcal{O}(d)$. The norm σ_p is one of the most important unitary invariant operator norms. Given a unitary invariant norm $\mathcal{N}$ in $\mathbb{R}^{d\times d}$ we can define the permutationally symmetric norm τ in $\mathbb{R}^d$ as follows,

$$\tau(x) := \mathcal{N}(D_x), \tag{3.21}$$

where D_x is the diagonal matrix with entries the coefficients of x. The unitary invariance of $\mathcal{N}$ implies that τ is permutationally symmetric. Given $T \in \mathbb{R}^{d\times d}$, since $(TT^*)^{\frac{1}{2}}$ is diagonalizable in an orthonormal basis we have that for every $T \in \mathbb{R}^{d\times d}$

$$\mathcal{N}(T) = \tau(s_1(T), \ldots, s_n(T)).$$

For example, if $\mathcal{N}$ is σ_p, τ is the usual ℓ_p-norm.

Set $\lambda(\tau) = \tau(\sum_{i=1}^d e_i)$ by Lemma 3.2.4, we have that

$$\frac{1}{\lambda(\tau)} B_\infty^d \subset B_\tau \subset \frac{d}{\lambda(\tau)} B_1^d,$$

and hence

$$\frac{1}{\lambda(\tau)} \mathcal{S}_\infty^d \subset B_{\mathcal{N}} \subset \frac{d}{\lambda(\tau)} \mathcal{S}_1^d. \tag{3.22}$$

Taking volumes we have that

$$\frac{1}{\lambda(\tau)} |\mathcal{S}_\infty^d|^{\frac{1}{d^2}} \leqslant |B_{\mathcal{N}}|^{\frac{1}{d^2}} \leqslant \frac{d}{\lambda(\tau)} |\mathcal{S}_1^d|^{\frac{1}{d^2}}.$$

In [Ray84], Saint Raymond computed the volume of $B_{\mathcal{S}_p^d}$ for $1 \leqslant p \leqslant \infty$, in particular, he proved that

$$|\mathcal{S}_\infty^d|^{\frac{1}{d^2}} \sim d|\mathcal{S}_1^d|^{\frac{1}{d^2}} \sim \frac{1}{\sqrt{d}}. \tag{3.23}$$

Hence , we conclude that

$$|B_{\mathcal{N}}|^{\frac{1}{d^2}} \sim |\frac{1}{\lambda(\tau)} \mathcal{S}_\infty^d|^{\frac{1}{d^2}}. \tag{3.24}$$

Theorem 3.3.6. *Let $B_{\mathcal{N}}$ be the unit ball of any unitary invariant norm $\mathcal{N}$ in $\mathbb{R}^{d\times d}$ and $L \subset \mathbb{R}^{d^2}$ a convex body in Rudelson's position, and let $A = (g_{ij})_{1\leqslant i,j\leqslant d^2} \in \mathbb{R}^{d^2\times d^2}$ be a random matrix with independent Gaussian entries $g_{ij} \sim \mathcal{N}(0,1)$. Then with probability greater than $1-2e^{-d}$, the body $\tilde{L} := \frac{AL}{\|A\|}\frac{1}{\tau(u)} \subset B_{\mathcal{N}}$ and also*

$$\frac{|B_{\mathcal{N}}|^{\frac{1}{d^2}}}{|\tilde{L}|^{\frac{1}{d^2}}} \leq d.$$

Proof of Theorem 3.3.6. Note that by equation (3.23) we know that

$$|B_{\mathcal{S}_\infty^d}|^{\frac{1}{d^2}} \sim d^{-\frac{1}{2}}.$$

On the other hand, set G a $d \times d$ matrix with independent Gaussian entries. By Gaussian Chevet's inequality (3.3.2), we know that

$$\mathbb{E}(\|G : \ell_2^d \to \ell_2^d\|) \leq \sqrt{d}.$$

Since $\|G : \ell_2^d \to \ell_2^d\|$ coincides with $\|G\|_{\mathcal{S}_\infty^d}$, we have that

$$\ell(B_{\mathcal{S}_\infty^d})|B_{\mathcal{S}_\infty^d}|^{\frac{1}{d^2}} \sim 1.$$

The inclusion $B_2^d \subset B_\infty^d$ implies $B_{\mathcal{S}_2^d} \subset B_{\mathcal{S}_\infty^d}$ and hence, $\|id : \mathcal{S}_2^d \to \mathcal{S}_\infty^d\| \leqslant 1$, which coincides with $\|id : \ell_2^{d^2} \to \mathcal{S}_\infty^d\| \leqslant 1$ since the norm in $\mathcal{S}_2^d$ is the Euclidean norm in $\mathbb{R}^{d\times d}$.

Using the fact that L is in Rudelson's position, by equation (3.19) we have that for $u = \sqrt{d}$, $A(L) \subset \|A\|\mathcal{S}_\infty^d$, and

$$\left(\frac{\|A\||\mathcal{S}_\infty^d|}{|A(L)|}\right)^{\frac{1}{d}} \leqslant d.$$

with probability greater than $1 - e^{-d}$.

By Equation (3.22),

$$\tilde{L} := \frac{1}{\lambda(\tau)}\frac{A(L)}{\|A\|} \subset \frac{1}{\lambda(\tau)}\mathcal{S}_\infty^d \subset B_{\mathcal{N}}$$

As $|\frac{1}{\lambda(\tau)}\mathcal{S}_\infty^d|^{\frac{1}{d^2}} \sim |B_{\mathcal{N}}|^{\frac{1}{d^2}}$ we obtain the desired bound. □

As a consequence of this we have the following corollary.

Corollary 3.3.7. *Let $\mathcal{N}$ be a unitary invariant norm in $\mathbb{R}^{d\times d}$ then*

$$\mathrm{lvr}(B_{\mathcal{N}}) \preceq d.$$

3.3.3 Tensor norms

Another natural class of convex bodies for which we can obtain sharp asymptotic bounds for the largest volume ratio are unit balls of norms given by tensor products of ℓ_p-spaces. Tensor products play a key role in the local theory of Banach spaces. They can be identified with natural spaces such as multilinear forms and homogeneous polynomials. We will review the basic definitions regarding tensor products, we refer to [DF92, Din99, Flo97] for a complete treatment on the subject. Given a normed space E we write $\bigotimes^m E$ for the *m-fold tensor product* of E, and $\bigotimes^{m,s} E$ stands for the *symmetric m-fold tensor product*, that is, the subspace of $\bigotimes^m E$ consisting of all tensor that can be written as $\sum_{i=1}^k \lambda_i \otimes^m x_i$, where $\lambda_i \in \mathbb{R}$ and $\otimes^m x_i = x_i \otimes \cdots \otimes x_i$. Observe that if E has dimension n, $\dim(\bigotimes^m E) = n^m$ and $\dim(\bigotimes^{m,s} E) = \binom{m+n-1}{n-1}$. Since we consider m as a fixed number, we have that in both cases the dimension of the space behaves like n^m.

There are many norms than can be defined on the tensor product, we will focus on two of them. The *projective tensor norm* is define as

$$\pi(x) = \inf\left\{\sum_{j=1}^r \prod_{i=1}^m \|x_i^r\|_E\right\},$$

where the infimum is taken over all representations of x, $x = \sum_{i=1}^{r} x_1 \otimes \cdots \otimes x_m$. The *injective tensor norm* is defined as

$$\varepsilon(x) = \sup \left| \sum_{j=1}^{r} \prod_{i=1}^{m} |\varphi_i(x_i^r)| \right|,$$

where the supremum runs over all $\varphi_1, \ldots, \varphi_m \in E^*$ and $\sum_{j=1}^{r} x_1 \otimes \cdots \otimes x_m$ is a fixed representation of x. Let $\alpha = \varepsilon$ or π, we write $\bigotimes_{\alpha}^{m}$ for m-fold product endowed with the norm α.

The space $\bigotimes_{\varepsilon}^{m} E$ can be identified with the space of m-linear operators defined on $(E^*)^m$ endowed with the usual supremum norm. An operator $T : E^m \to \mathbb{R}$ is *m-nuclear* if can be written as,

$$T = \sum_{i=1}^{\infty} \varphi_1^i \ldots \varphi_m^i,$$

with $\varphi \in E^*$ and $\sum_{i=1}^{\infty} \|\varphi_1^i\|_{E^*} \ldots \|\varphi_m^i\|_{E^*} < \infty$. We can define the following norm on the space of all m-nuclear operators.

$$\|T\|_{nuc} = \inf\{\sum_{i=1}^{\infty} \|\varphi_1^i\|_{E^*} \ldots \|\varphi_m^i\|_{E^*}\},$$

where infimum is taken over all representation of T as above. With this norm, the space of all m-nuclear operators on $(E^*)^n$ can be identified with $\bigotimes_{\pi}^{m} E$.

In the same way we can define the corresponding injective and projective norms in the symmetric tensor product. We define the *symmetric projective norm* as,

$$\pi_s(x) := \inf \left\{ \sum_{i=1}^{r} \|x_i\|_E^m \right\},$$

where the infimum is taken over all the representation of x of the form $x = \sum_{i=1}^{r} \otimes^m x_i$.

The *symmetric injective norm* is computed as follows,

$$\varepsilon_s(x) = \sup_{\varphi \in B_{E'}} \left| \sum_{i=1}^{r} \varphi(x_i)^m \right|,$$

where $x = \sum_{i=1}^{r} \otimes^m x_i$ is a fixed representation of x. It is important to notice that, in general, the tensor norms and their symmetric version do not coincide. That is, $\alpha\big|_{\bigotimes^{m,s} E} \neq \alpha_s$.

The spaces $\bigotimes_{\varepsilon_s}^{m,s} E$ and $\bigotimes_{\pi_s}^{m,s} E$ can be represented as spaces of polynomials. A function $p : X \to \mathbb{R}$ is said to be an *m-homogeneous polynomial* if there is an m-linear form $\varphi : E^m \to \mathbb{R}$ such that $p(x) = \varphi(x, \dots, x)$. We write $\mathcal{P}(^m E)$ for the set of m-homogeneous polynomials on E. If we define in $\mathcal{P}(^m E^*)$ the norm,

$$\|p\| := sup_{x \in B_E} |p(x)|,$$

the space is isometric to $\bigotimes_{\varepsilon_s}^{m,s} E$.

An m-homogeneous polynomial is said to be *nuclear* if can be written as

$$p(x) = \sum_{i=1}^{\infty} \lambda_i (\varphi_i(x))^m,$$

where $\lambda_i \in \mathbb{R}$, $\varphi_i \in E^*$ and $\sum_{i=1}^{\infty} |\lambda_i| \|\varphi_i\|_{E^*} < \infty$. We write $\mathcal{P}_{nuc}(^m E)$ for the space of nuclear polynomials. If we define the norm,

$$\|p\|_{nuc} = \inf \left\{ \sum_{i=1}^{\infty} |\lambda_i| \|\varphi_i\|_{E^*} \right\},$$

where the infimum is taken over all representations of p as above. The space $\mathcal{P}_{nuc}(^m E^*)$ with the correspondent norm is identified with $\bigotimes_{\pi_s}^{m,s} E$.

We are going to work with $E = \ell_p^n$ and, as we did in the case of the unitary invariant norms, in order to obtain bounds for the volume ratio we need to have estimates of some geometrical parameters of the involved spaces. Defant and Prengel [DP09] obtained asymptotic estimates for many of them. We summarize their results in the next proposition.

Proposition 3.3.8. *For $m \in \mathbb{N}$ set $d = n^m$ and $d_s = \binom{m+n-1}{n-1}$. For each $1 \leqslant p \leqslant \infty$ we have,*

1. $\left|B_{\bigotimes_{\varepsilon_s}^{m,s} \ell_p^n}\right|^{\frac{1}{d_s}} \sim \left|B_{\bigotimes_{\varepsilon}^{m} \ell_p^n}\right|^{\frac{1}{d}} \sim \begin{cases} n^{m(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}} & p \leqslant 2 \\ n^{-\frac{1}{p}} & p \geqslant 2. \end{cases}$

2. $\left|B_{\bigotimes_{\pi_s}^{m,s} \ell_p^n}\right|^{\frac{1}{d_s}} \sim \left|B_{\bigotimes_{\pi}^{m} \ell_p^n}\right|^{\frac{1}{d}} \sim \begin{cases} n^{1-\frac{1}{p}-m} & p \leqslant 2 \\ n^{\frac{1}{2}-m(\frac{1}{2}+\frac{1}{p})} & p \geqslant 2. \end{cases}$

3. $\ell(B_{\bigotimes_{\varepsilon_s}^{m,s} \ell_p^n}) \sim \ell(B_{\bigotimes_{\varepsilon}^{m} \ell_p^n}) \sim \begin{cases} n^{m(\frac{1}{p}-\frac{1}{2})+\frac{1}{2}} & p \leqslant 2 \\ n^{\frac{1}{p}} & p \geqslant 2. \end{cases}$

4. $\ell(B_{\bigotimes_{\pi_s}^{m,s} \ell_p^n}) \sim \ell(B_{\bigotimes_{\pi}^{m} \ell_p^n}) \sim \begin{cases} n^{m-1+\frac{1}{p}} & p \leqslant 2 \\ n^{m(\frac{1}{2}+\frac{1}{p})-\frac{1}{2}} & p \geqslant 2. \end{cases}$

5. $\|id : \ell_2^{d_s} \to \bigotimes_{\varepsilon_s}^{m,s} \ell_p^n\| \sim \|id : \ell_2^{d} \to \bigotimes_{\varepsilon}^{m} \ell_p^n\| \sim \begin{cases} n^{m(\frac{1}{2}-\frac{1}{p})} & p \leqslant 2 \\ 1 & p \geqslant 2. \end{cases}$

6. $\|id : \ell_2^{d_s} \to \bigotimes_{\pi_s}^{m,s} \ell_p^n\| \sim \|id : \ell_2^d \to \bigotimes_{\pi}^m \ell_p^n\| \sim \begin{cases} n^{\frac{m}{2}+\frac{1}{p}-1} & p \leqslant 2 \\ n^{\frac{m}{p}-\frac{1}{2}} & 2 \leqslant p \leqslant 2m \\ 1 & p \geqslant 2m. \end{cases}$

All the proofs can be found in [DP09]. The comparison between the full and symmetric tensor products follows from [DP09, Proposition 3.1]. The estimates (1) and (2) are in [DP09, Theorem 4.2]. For (3) and (4) see [DP09, Lemma 4.3]. The proof of (5) follows form the fact that

$$\|id : \ell_2^{n^m} \to \bigotimes_{\varepsilon}^{m} \ell_p^n\| = \|id : \ell_2^n \to \ell_p^n\|^m.$$

For (6) more technical arguments are required, the result is stated in [DP09, Lemma 5.2].

Observe that, in particular, for every space E involved in the last proposition, we have that

$$\ell(B_E)|B_E|^{\frac{1}{\dim(E)}} \sim 1.$$

Hence, by equation (3.19), if $K = B_E$, and $N = \dim(E)$ we have, that if L is any convex body in Rudelson's position and A is a random Gaussian matrix,

$$\left(\frac{\|A\||K|}{|A(L)|}\right)^{\frac{1}{N}} \preceq \sqrt{N} + (\log N + u)\sqrt{N}\|id : \ell_2^N \to E\||B_E|^{\frac{1}{N}}.$$

with probability greater than $1 - 2e^{-u^2}$. Now, if we take for example, $E = \bigotimes_{\varepsilon}^m \ell_p^n$ with $p \leqslant 2$, we have that

$$\|id : \ell_2^N \to \bigotimes_{\varepsilon}^{m} \ell_p^n\||B_{\bigotimes_{\varepsilon}^m \ell_p^n}|^{\frac{1}{N}} = n^{2m(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}.$$

So, taking $u = n^{-2m(\frac{1}{2}-\frac{1}{p})+\frac{1}{2}} \geqslant \log(N)$, we get

$$\left(\frac{\|A\||K|}{|A(L)|}\right)^{\frac{1}{N}} \preceq \sqrt{N}.$$

It can be checked that in all cases, $\|id : \ell_2^N \to E\||B_E|^{\frac{1}{N}} \preceq \frac{1}{\log(N)}$. So, choosing

$$u^{-1} = \|id : \ell_2^N \to E\||B_E|^{\frac{1}{N}}$$

we have that with high probabilty

$$\left(\frac{\|A\||K|}{|A(L)|}\right)^{\frac{1}{N}} \preceq \sqrt{N}.$$

Arguing analogously for the other cases we obtain the following theorem.

Theorem 3.3.9. *For $E = \bigotimes_{\varepsilon}^m \ell_p^n, \bigotimes_{\varepsilon_s}^{m,s} \ell_p^n, \bigotimes_{\pi}^m \ell_p^n$ or $\bigotimes_{\pi_s}^{m,s} \ell_p^n$ and $N = dim(E)$ we have that,*

$$\mathrm{lvr}(B_E) \leqslant \sqrt{N}.$$

3.4 Gaussian processes and Chevet's inequality

We will now present the proof of Proposition 3.3.3. For that we need to introduce some definitions regarding random processes. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. By a *random process* we simply mean a family of random variables $\{X_t\}_{t\in T} : \Omega \to \mathbb{R}$ where T is any set. The problem of bounding the norm of an operator can be thought as the problem of bounding the supremum of a random process. Since the supremum of measurable functions is not necessarily measurable, we define the supremum of a process as follows:

$$\sup_{t\in T} X_t = \sup_{\substack{T_0\subset T,\\ T_0 \text{ finite}}} (\sup_{t\in T_0} X_t).$$

We define the *increment* of the process as

$$d(s,t) := \|X_s - X_t\|_2 = (\mathbb{E}(X_s - X_t)^2)^{\frac{1}{2}}.$$

We say that the process is a *Gaussian process* if for every finite set $T_0 \subset T$, the vector $(X_i)_{i\in T^0}$ has normal distribution. Equivalently, any linear combination, $\sum a_t X_t$ is a normal random variable. A basic example is the so-called *canonical Gaussian process*

$$X_t := \langle g, t\rangle, \tag{3.25}$$

where g is a standard Gaussian random vector and $T \subset \mathbb{R}^n$ is any set. In this case the increments coincide with the euclidean distances in $\mathbb{R}^n$.

The proof of Chevet's inequality relies on an inequality of Sudakov (see for example [AAGM15, Proposition 9.1.7]) that compares the supremum of two Gaussian process that have comparable increments.

Proposition 3.4.1 (Sudakov)**.** *Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and let $\{X_t\}_{t\in T}$ and $\{Y_t\}_{t\in T}$ be two Gaussian process, with $\mathbb{E}(X_t) = \mathbb{E}(Y_t) = 0$ for all $t \in T$. If*

$$\|X_t - X_s\|_2 \leqslant \|Y_t - Y_s\|_2 \tag{3.26}$$

for every $s, t \in T$, then

$$\mathbb{E}\sup_{t\in T} X_t \leqslant \mathbb{E}\sup_{t\in T} Y_t.$$

In order two deduce Chevet's inequality from Proposition 3.4.1, set $T = L \times K^\circ$ and consider the Gaussian process given by

$$X_{(x,y^*)} := \langle Ax, y^*\rangle,$$

where A is a matrix with entries $g_{ij} \sim \mathcal{N}(0,1)$ and

$$Y_{(x,y^*)} := \langle g, x\rangle \|id : \ell_2^n \to X_K\| + \langle h, y^*\rangle \|id : \ell_2^n \to X_{L^\circ}\|,$$

where $g = (g_1, \dots, g_n)$, $h = (h_1, \dots, h_n)$ and $(g_i)_{i=1}^n, (h_j)_{j=1}^n$ are independent standard Gaussian variables. Note that

$$\mathbb{E}(\sup_{(x,y^*)\in T} Y_{(x,y^*)}) = (\ell(K)\|id : \ell_2^n \to X_{L^\circ}\| + \|id : \ell_2^n \to X_K\|\ell(L^\circ)).$$

In order to prove Proposition 3.3.3, we are going to use a result of Talagrand ([Ver18, Theorem 8.5.5]) that is set in a more general context. In our setting it could be seen as a tail bound for Proposition 3.4.1. Here, $\|\cdot\|_{\psi_2}$ stands for the sub-Gaussian norm, defined as

$$\|X\|_{\psi_2} := \inf\{\lambda > 0 \mid \int_\Omega e^{\frac{|X(\omega)|^2}{\lambda^2}} d\mu \leqslant 2\}.$$

Theorem 3.4.2. *Let $(X_t)_{t\in T}$ be a random process and $(Y_t)_{t\in T}$ a Gaussian process such that $\|X_t - X_s\|_{\psi_2} \leqslant \|Y_t - Y_s\|_2$. Then, for every $u \geqslant 0$, the event*

$$\sup_{t\in T} |X_t| \preceq (\mathbb{E}(\sup Y_t) + u\,\mathrm{diam}(T))$$

holds with probability at least $1 - 2e^{-u^2}$. Here the diameter of T is with respect to the distance defined by the increments of the process Y_t.

We now include a sketch of the proof of Proposition 3.3.3.

Sketch of the proof of Proposition 3.3.3. As before, we define random process in $L \times K^\circ$ given by

$$X_{(x,y^*)} := \langle Ax, y^*\rangle.$$

Note that if we consider the Gaussian process

$$Y_{(x,y^*)} := \langle g, x\rangle\|id : \ell_2^n \to X_K\| + \langle h, y^*\rangle\|id : \ell_2^n \to X_{L^\circ}\|,$$

where $g = (g_1, \dots, g_n)$, $h = (h_1, \dots, h_n)$ and $(g_i)_{i=1}^n, (h_j)_{j=1}^n$ are independent standard Gaussian variables; we have,

$$\|X_{(x,y^*)} - X_{(\tilde{x},\tilde{y}^*)}\|_{\psi_2} \preceq \|Y_{(x,y^*)} - Y_{(\tilde{x},\tilde{y}^*)}\|_2.$$

In fact,

$$\begin{aligned}
\|X_{(x,y^*)} - X_{(\tilde{x},\tilde{y}^*)}\|_{\psi_2} &= \|\sum_{i,j} g_{ij}(x_i y_j^* - \tilde{x}_i \tilde{y}_j^*)\|_{\psi_2} \\
&\leqslant \left(\sum_{i,j} \|g_{ij}(x_i y_j^* - \tilde{x}_i \tilde{y}_j^*)\|_{\psi_2}^2\right)^{\frac{1}{2}} \\
&\leqslant \left(\sum_{i,j} |x_i y_j^* - \tilde{x}_i \tilde{y}_j^*|^2\right)^{\frac{1}{2}} \\
&\leqslant \left(\sum_{i,j} |x_i y_j^* - \tilde{x}_i y_j^* + \tilde{x}_i y_j^* - \tilde{x}_i \tilde{y}_j^*|_2^2\right)^{\frac{1}{2}} \\
&\leqslant \|x - \tilde{x}\|_2 \|\tilde{y}^*\|_2 + \|x\|_2 \|y^* - \tilde{y}^*\|_2 \\
&\leqslant \|x - \tilde{x}\|_2 \|id : \ell_2^n \to X_K\| + \|y^* - \tilde{y}^*\|_2 \|id : \ell_2^n \to X_{L^\circ}\|. \\
&= \|Y_{(x,y^*)} - Y_{(\tilde{x},\tilde{y}^*)}\|_2.
\end{aligned}$$

Applying Theorem 3.4.2 we get,

$$\begin{aligned}
\|A : X_L \to X_K\| &= \sup_{(x,y^*)\in L\times K^\circ} X_{(x,y^*)} \\
&\preceq \left(\mathbb{E}[\sup_{(x,y^*)\in L\times K^\circ} Y_{(x,y^*)}] + u \ \operatorname{diam}(K \times L^\circ)\right),
\end{aligned}$$

with probability at least $1 - e^{-u^2}$.

The result follows from the fact that

$$\mathbb{E}[\sup_{(x,y^*)\in L\times K^\circ} Y_{(x,y^*)}] = \ell(K)\|id : \ell_2^n \to X_{L^\circ}\| + \ell(L^\circ)\|id : \ell_2^n \to X_K\|$$

and $\operatorname{diam}(L \times K^\circ) \sim \|id : \ell_2^n \to X_{L^\circ}\| \cdot \|id : \ell_2^n \to X_K\|$. $\square$

Chapter 4

Lower bounds

In this chapter we deal with lower bounds for the largest volume ratio. We will prove that for every convex body $K \subset \mathbb{R}^n$, $\operatorname{lvr}(K) \geq \sqrt{n}$. If we combine this bound with the one obtained in the previous chapter we conclude that it is the best possible general bound. The key ingredient for the proof is the use of certain random polytopes that were introduced by Gluskin while studying the diameter of the Banach-Mazur compactum. We define this polytopes and show how to use them to bound the volume ratio in Section 4.2.

4.1 Lower bound for the largest volume ratio

We now treat lower bounds for the largest volume ratio of a given convex body K. That is, for a convex body K we want to find another body, L, such that $\operatorname{vr}(K, L)$ is "large". Khrabrov, in [Khr01], proved that for every convex body $K \subset \mathbb{R}^n$ there is a convex body L such that

$$\operatorname{vr}(K, L) \geq \sqrt{\frac{n}{\log \log(n)}}. \tag{4.1}$$

We will remove the double logarithm in equation (4.1) proving that

$$\operatorname{lvr}(K) \geq \sqrt{n} \tag{4.2}$$

holds for every convex body $K \subset \mathbb{R}^n$. Taking into account that, as we have seen in the previous chapter, for many collections of bodies we know that $\operatorname{lvr}(K) \preceq \sqrt{n}$, the bound in (4.2) is the best possible general bound. By Proposition 3.1.1, he have

$$\operatorname{vr}(K - K, L) \leqslant \operatorname{vr}(K, L) \operatorname{vr}(K - K, K) \sim \operatorname{vr}(K, L).$$

Hence, we can reduce the problem of bounding from bellow the largest volume ratio to the centrally symmetric case. Recall the statement of Proposition 3.1.2 (1): given $K, L \subset \mathbb{R}^n$ two centrally symmetric convex bodies,

$$\mathrm{vr}(K, L) = \left(\frac{|K|}{|L|}\right)^{\frac{1}{n}} \cdot \inf_{T \in SL(n,\mathbb{R})} \|T : X_L \to X_K\|. \tag{4.3}$$

Therefore, to show "good" lower bounds for $\mathrm{lvr}(K)$ (for K centrally symmetric) we need a body L such that its volume is "small" and the norm $\|T : X_L \to X_K\|$ is large for every operator $T \in SL(n, \mathbb{R})$. Recall that, by equation (3.4) $\mathrm{lvr}(K)$ can be computed taking the supremum of $\mathrm{vr}(K, L)$ only over centrally symmetric bodies L.

4.2 Gluskin's polytopes

The key idea of [Khr01] is to use the probabilistic method. He considered random polytopes with vertices distributed on the unit sphere and proved that the probability that such a body satisfies the bound (4.1) is positive. He is based on Gluskin's work [Glu81], who defined the random bodies

$$L^{(m)} := \mathrm{absconv}\{X_1, \ldots, X_m, e_1, \ldots, e_n\}, \tag{4.4}$$

where $\{X_i\}_{i=1}^m$ are independent vectors distributed according to the normalized Haar measure in S^{n-1}, σ. We can construct σ as a cone measure as follows:

$$\sigma_n(A) = \frac{|\{x \in B_2^n \mid \frac{x}{\|x\|_2} \in A\}|}{|B_2^n|}, \tag{4.5}$$

for any set $A \subset S^{n-1}$.

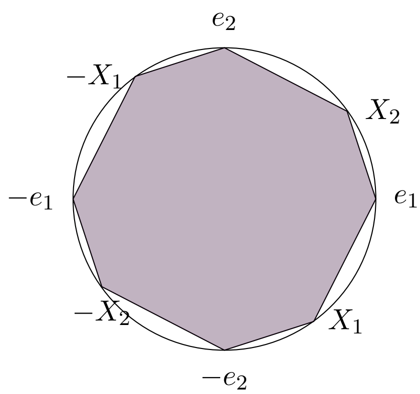

Figure 4.1: Random polytope $L^{(2)}$ in $\mathbb{R}^2$.

He used this construction to find the asymptotic order of the diameter of the *Banach-Mazur compactum* (also known as Minkowski compactum),

$\mathcal{C}_n$, the set of all n-dimensional centrally symmetric convex bodies endowed with the *Banach-Mazur distance*:

$$d_{BM}(K,L) = \inf\{a \cdot b \mid \frac{1}{a}K \subset TD \subset bK\}, \tag{4.6}$$

where the infimum is taken over all invertible linear operators $T : \mathbb{R}^n \to \mathbb{R}^n$. John's theorem implies that $d_{BM}(K, B_2^n) \leqslant \sqrt{n}$ for every body K, and hence $\operatorname{diam}(\mathcal{C}_n) \leqslant n$. Gluskin showed that, if $m \sim n$, with positive probability, two random polytopes $L^{(m)}$ and $L'^{(m)}$ fulfill:

$$d_{BM}(L^{(m)}, L'^{(m)}) \geq n.$$

It should be noted that, since $L^{(m)} \subset B_2^n$, by Lemma 3.2.1 the volume of the random polytope $L^{(m)}$ is bounded by

$$|L^{(m)}|^{\frac{1}{n}} \leq \frac{\sqrt{\log(\frac{m}{n})}}{n}. \tag{4.7}$$

In fact, this bound is the exact asymptotic growth of $|L^{(m)}|^{\frac{1}{n}}$ with probability greater than or equal to $1 - \frac{1}{m}$ [BGVV14, Chapter 11].

If we combine the volume bound (4.7) with (4.3) we need to prove the existence of an operator T for which we can bound from below the norm $\|T : X_{L^{(m)}} \to X_K\|$ for all random polytopes. Note that as m grows, $\inf_{T \in SL(n,\mathbb{R})} \|T : X_{L^{(m)}} \to X_K\|$ becomes larger but $\frac{1}{|L^{(m)}|^{1/n}}$ decreases, so there is some sort of trade-off.

In [Khr01], for $m = n\log(n)$, it is shown that, with high probability, the norm $\|T : X_{L^{(m)}} \to X_K\|$ is "large" for every $T \in SL(n,\mathbb{R})$. To achieve all this he proved the following interesting inequality:

If $K \subset \mathbb{R}^n$ is in Löwner's position then for very $m \in \mathbb{N}$ and every $\beta > 0$,

$$\mathbb{P}\left\{ \text{There exists } T \in SL(n,\mathbb{R}) : \|T : X_{L^{(m)}} \to X_K\| \leqslant \beta \left(\frac{|B_2^n|}{|K|}\right)^{1/n} \right\} \tag{4.8}$$
$$\leqslant \left(C\sqrt{n}\right)^{n^2} \left(\frac{|B_2^n|}{|K|}\right)^n \beta^{nm-n^2}.$$

In order to prove main contribution of this section, Theorem 4.2.9, we present the following refinement of the previous estimate.

Proposition 4.2.1. *Let $K \subset \mathbb{R}^n$ be centrally symmetric convex body and $L^{(m)}$ the random polytope defined in* (4.4)*, then for every $\beta > 0$ we have*

$$\mathbb{P}\left\{ \text{There exists } T \in SL(n,\mathbb{R}) : \|T : X_{L^{(m)}} \to X_K\| \leqslant \beta \left(\frac{|B_2^n|}{|K|}\right)^{1/n} \right\}$$
$$\leqslant C^{n^2} \left(\|id : \ell_2^n \to X_K\|\sqrt{n}|K|^{\frac{1}{n}}\right)^{n^2} (2\beta)^{nm},$$

for some absolute constant $C > 0$.

To prove Proposition 4.2.1 we need a couple of lemmas. The first one, Lemma 4.2.2 bounds the probability that a fixed operator $T : \mathbb{R}^n \to \mathbb{R}^n$ of determinant one has "large" norm. The second one, Lemma 4.2.3 is a technical tool which bounds the number of points in an ε-net for an adequate set. This allows us to use a standard ε-net argument to pass from one fixed operator to all operators in $SL(n, \mathbb{R})$.

Lemma 4.2.2. *Let $K \subset \mathbb{R}^n$ be a convex body, $L^{(m)}$ the random polytope in (4.4), $T \in SL(n, \mathbb{R})$ and $\alpha > 0$. Then*

$$\mathbb{P}\left\{\|T : X_{L^{(m)}} \to X_K\| \leqslant \alpha\right\} \leqslant \alpha^{mn} \left(\frac{|K|}{|B_2^n|}\right)^m. \tag{4.9}$$

Proof. Note that if $L^{(m)} = \text{absconv}\{X_1, \dots, X_m, e_1, \dots, e_n\}$, in order to have $\|T : X_{L^{(m)}} \to X_K\| \leqslant \alpha$, we must have that for all $1 \leqslant i \leqslant m$, $TX_i \in \alpha K$. Hence,

$$\begin{aligned}\mathbb{P}\left\{\|T : X_{L^{(m)}} \to X_K\| \leqslant \alpha\right\} &\leqslant \mathbb{P}\{\text{ For all } 1 \leqslant i \leqslant m,\ TX_i \in \alpha K\} \\ &= \sigma\{S^{n-1} \textstyle\bigcap T^{-1}(\alpha K)\}^m,\end{aligned}$$

Recall that σ can be obtained as a cone measure (equation (4.5)) and hence,

$$\sigma\{S^{n-1} \textstyle\bigcap T^{-1}(\alpha K)\} = \dfrac{|\{x \in B_2^n \mid \frac{x}{\|x\|_2} \in S^{n-1} \bigcap \alpha T^{-1}(K)\}|}{|B_2^n|}.$$

Since $\alpha T^{-1}(K)$ is a convex body which contains the origin, we have that

$$\{x \in B_2^n \mid \frac{x}{\|x\|_2} \alpha T^{-1}(K)\} \subset \alpha T^{-1}(K),$$

and hence,

$$\begin{aligned}\sigma\{S^{n-1} \textstyle\bigcap T^{-1}(\alpha K)\} &\leqslant \dfrac{|\alpha T^{-1}(K)|}{|B_2^n|} \\ &= \alpha^n \dfrac{|K|}{|B_2^n|}.\end{aligned}$$

□

We now present the second lemma involved in the proof of Proposition 4.2.1. This should be compared with [Khr01, Lemma 5]: note that the set and the metric differ. This subtle but important modification is the key ingredient we need.

Lemma 4.2.3. *Let $K \subset \mathbb{R}^n$ be a convex body, $\gamma > 0$ and*

$$\mathcal{M}_\gamma^K := \{T \in SL(n,\mathbb{R}) \text{ and } \|T : \ell_1^n \to X_K\| \leqslant \gamma\}\,.$$

There is a γ-net, $\mathcal{N}_\gamma^K$ for $\mathcal{M}^K$ in the metric $\mathcal{L}(\ell_2^n, X_K)$ such that

$$\#\mathcal{N}_\gamma^K \leqslant C^{n^2}\left(\|id : \ell_2^n \to X_K\|\sqrt{n}|K|^{1/n}\right)^{n^2}.$$

Proof. Let U be the unit ball of $\mathcal{L}(\ell_2^n, X_K)$. By the standard identification we consider $\mathcal{M}^K$ and U as subsets of $\mathbb{R}^{n\times n}$. Let $\mathcal{N}_\gamma^K$ be a maximal collection of elements of $\mathcal{M}^K$ γ-separated. These elements form an γ-net and, for every $\xi \in \mathcal{N}_\gamma^K$, the balls $\xi + \frac{\gamma}{2}U$ are disjoints. Since

$$\|T : \ell_1^n \to X_K\| \leqslant \|T : \ell_2^n \to X_K\|,$$

we have that $\gamma U \subset \{T : \|T : \ell_1^n \to X_K\| \leqslant \gamma\}$ and then

$$\bigcup_{\xi\in\mathcal{N}_\gamma^K} \xi + \frac{\gamma}{2}U \subset \frac{3}{2}\{T : \|T : \ell_1^n \to X_K\| \leqslant \gamma\}\,.$$

Computing the volume on both sides, we get the following bound for $\#\mathcal{N}_\varepsilon^K$,

$$\#\mathcal{N}_\gamma^K\left(\frac{\gamma}{2}\right)^{n^2}|U| \leqslant \left(\frac{3}{2}\right)^{n^2}|\{T : \|T : \ell_1^n \to X_K\| \leqslant \gamma\}|$$

$$\#\mathcal{N}_\gamma^K \leqslant \left(\frac{3}{\gamma}\right)^{n^2}\frac{|\{T : \|T : \ell_1^n \to X_K\| \leqslant \gamma\}|}{|U|}. \tag{4.10}$$

Now notice that

$$\begin{aligned}\{T \in \mathcal{L}(\ell_1^n, X_K) : \|T : \ell_1^n \to X_K\| \leqslant \gamma\}&\\ \subset \left\{X \in \mathbb{R}^{n\times n} : X_i \in \gamma\cdot K \text{ for all } i\right\}&\\ \subset \underbrace{(\gamma K)\times\cdots\times(\gamma K)}_{n}&,\end{aligned} \tag{4.11}$$

and hence

$$|\{T : \|T : \ell_1^n \to X_K\| \leqslant \gamma\}| \leqslant (\gamma)^{n^2}|K|^n. \tag{4.12}$$

In order to bound equation (4.10) we need a lower bound for $|U|$. By passing to spherical coordinates it can be checked that

$$\frac{|U|}{|B_2^{n^2}|} = \int_{S^{n^2-1}} \|T\|_{\mathcal{L}(\ell_2^n, X_K)}^{-n^2} d\sigma(T), \tag{4.13}$$

where σ is the normalized Haar measure on S^{n^2-1}. Now we apply Hölder's inequality to get

$$1 \leqslant \left(\int\limits_{S^{n^2-1}} \|T\|^2_{\mathcal{L}(\ell_2^n, X_K)} d\sigma(T)\right)^{1/2} \left(\int\limits_{S^{n^2-1}} \|T\|^{-2}_{\mathcal{L}(\ell_2^n, X_K)} d\sigma(T)\right)^{1/2}$$

$$\leqslant \left(\int\limits_{S^{n^2-1}} \|T\|^2_{\mathcal{L}(\ell_2^n, X_K)} d\sigma(T)\right)^{1/2} \left(\int\limits_{S^{n^2-1}} \|T\|^{-n^2}_{\mathcal{L}(\ell_2^n, X_K)} d\sigma(T)\right)^{1/n^2}.$$

Therefore,

$$\frac{|U|}{|B_2^{n^2}|} \geqslant \left(\int\limits_{S^{n^2-1}} \|T\|^2_{\mathcal{L}(\ell_2^n, X_K)} d\sigma(T)\right)^{-n^2/2}.$$

By comparing spherical and Gaussian means (equation (1.10)) and applying Gaussian Chevet's inequality 3.3.2 (recall that all Gaussian moments are comparable (1.12)), we have that

$$\left(\int\limits_{S^{n^2-1}} \|T\|^2_{\mathcal{L}(\ell_2^n, X_K)} d\sigma(T)\right)^{1/2} \preceq \frac{1}{n}\left(\ell(K) + \|id : \ell_2^n \to X_K\|\sqrt{n}\right),$$

which implies

$$\left(\ell(K) + \|id : \ell_2^n \to X_K\|\sqrt{n}\right)^{-n^2} C^{-n^2} \geqslant |U|. \tag{4.14}$$

Using (4.12) and (4.14) in Equation (4.10) we obtain:

$$\#\mathcal{N}_\gamma^K \leqslant C^{n^2}\left(\ell(K)|K|^{1/n} + \|id : \ell_2^n \to X_K\|\sqrt{n}|K|^{1/n}\right)^{n^2}.$$

Now notice that since $B_2^n \subset \|id : \ell_2^n \to X_K\|K$ we have that,

$$\frac{1}{\|id : \ell_2^n \to X_K\|} K^\circ \subset B_2^n$$

and hence $\omega(K^\circ) \leqslant \|id : \ell_2^n \to X_K\|$. Recalling that $\ell(K) \sim \sqrt{n}\omega(K^\circ)$ we get,

$$\ell(K) \leqslant \|id : \ell_2^n \to X_K\|\sqrt{n}, \tag{4.15}$$

what concludes the proof. ☐

Now we present the proof of Proposition 4.2.1.

Proof of Proposition 4.2.1. Let $\{X_i\}_{i=1}^m \subset S^{n-1}$ and $L^{(m)}$ be the polytope in (4.4) such that there exists $T \in SL(n,\mathbb{R})$ with $\|T : X_{L^{(m)}} \to X_K\| \leqslant \gamma$. As $\ell_1^n \subset L^{(m)}$, T lies in the set $\mathcal{M}^K$ defined in Lemma 4.2.3. Consider a γ-net, $\mathcal{N}_\gamma^K$ for $\mathcal{M}^K$ for the metric $\mathcal{L}(\ell_2^n, X_K)$ such that

$$\#\mathcal{N}_\gamma^K \leqslant C^{n^2}\left(\|id : \ell_2^n \to X_K\|\sqrt{n}|K|^{\frac{1}{n}}\right)^{n^2}. \tag{4.16}$$

Let $S \in \mathcal{N}_\gamma^K$ such that $\|S - T\|_{\mathcal{L}(\ell_2^n, X_K)} \leqslant \gamma$, then

$$\begin{aligned}\|S : X_{L^{(m)}} \to X_K\| &\leqslant \|T : X_{L^{(m)}} \to X_K\| + \|S - T : X_{L^{(m)}} \to X_K\| \\ &\leqslant \gamma + \|S - T : \ell_2^n \to X_K\| \\ &\leqslant 2\gamma,\end{aligned}$$

where we have used the fact that $\|S-T : X_{L^{(m)}} \to X_K\| \leqslant \|S-T : \ell_2^n \to X_K\|$ since, by construction, $L^{(m)} \subset B_2^n$. Hence,

$$\begin{aligned}\mathcal{B}_\gamma := \{\text{There exists } T \in SL(n,\mathbb{R}) : \|T : X_{L^{(m)}} \to X_K\| \leqslant \gamma\} \\ \subset \bigcup_{S \in \mathcal{N}_\gamma^K} \{\|S : X_{L^{(m)}} \to X_K\| \leqslant 2\gamma\}.\end{aligned}$$

Take $\gamma_0 := \beta\left(\frac{|B_2^n|}{|K|}\right)^{\frac{1}{n}}$, by the union bound, equation (4.16) and Lemma 4.2.2

$$\mathbb{P}(\mathcal{B}_{\gamma_0}) \leqslant C^{n^2}\left(\|id : \ell_2^n \to X_K\|\sqrt{n}|K|^{\frac{1}{n}}\right)^{n^2}(2\beta)^{nm}, \tag{4.17}$$

which concludes the proof. □

As a consequence of Proposition 4.2.1 we obtain the following result.

Proposition 4.2.4. *Let $K \subset \mathbb{R}^n$ be centrally symmetric convex body such that*

$$\|id : \ell_2^n \to X_K\|\sqrt{n}|K|^{\frac{1}{n}} \sim 1.$$

Given $\delta \geqslant 1$, with probability greater than or equal to $1 - e^{-n^2}$ the random polytope $L^{(\lceil \delta n \rceil)}$ in (4.4) verifies

$$\sqrt{n} \preceq \mathrm{vr}(K, L^{(\lceil \delta n \rceil)}).$$

In particular, $\sqrt{n} \preceq \mathrm{lvr}(K)$.

Proof. By Proposition 4.2.1 we know that there is an absolute constant $C > 0$ such that, for every $\beta > 0$,

$$\mathbb{P}\left\{\text{There exists } T \in SL(n,\mathbb{R}) : \|T : X_{L^{(m)}} \to X_K\| \leqslant \beta \left(\frac{|B_2^n|}{|K|}\right)^{1/n}\right\}$$
$$\leqslant C^{n^2}(2\beta)^{nm}.$$

If $m = \lceil \delta n \rceil$ and $\beta \leqslant \frac{1}{2}(Ce)^{-\frac{1}{\delta}}$, then with probability at least $1 - e^{-n^2}$ the random polytope verifies

$$\|T : X_{L^{(\lceil \delta n \rceil)}} \to X_K\| \geqslant \beta \left(\frac{|B_2^n|}{|K|}\right)^{1/n} \sim \frac{1}{\sqrt{n}|K|^{\frac{1}{n}}}, \tag{4.18}$$

for every $T \in SL(n,\mathbb{R})$.

Hence, by Equations (4.7) and (4.18) and Proposition 3.1.2 (1) we have

$$\sqrt{n} \preceq \mathrm{vr}(K, L^{(\lceil \delta n \rceil)}),$$

which concludes the proof. □

In order to prove Theorem 4.2.9 we will show that any given convex body can be approximated by another one which fulfils the hypothesis of the previous proposition. To achieve this we will make use of two deep and important results in the theory for isotropic convex bodies: Paouris' result on the concentration of mass and Klartag's perturbation with uniformly bounded isotropic constant (also known as Klartag's solution to the isomorphic slicing problem).

Theorem 4.2.5 ([Pao06],Theorem 1.1)**.** *There is an absolute constant $c > 0$ such that if $K \subset \mathbb{R}^n$ is an isotropic convex body, then*

$$\mathbb{P}\{x \in K \ : \ \|x\|_2 \geqslant cL_K\sqrt{n}t\} \leqslant e^{-\sqrt{n}t}$$

for every $t \geqslant 1$.

Theorem 4.2.6 ([Kla06], Theorem 1.1)**.** *Let $K \subset \mathbb{R}^n$ be a convex body and let $\varepsilon > 0$. Then there is a convex body $T \subset \mathbb{R}^n$ such that*

1. *$d(K,T) < 1 + \varepsilon$,*

2. *$L_T < \frac{c}{\sqrt{\varepsilon}}$.*

Here $c > 0$ is an absolute constant and

$$d(K,T) = \inf\{ab : a, b > 0, \exists\, x, y \in \mathbb{R}^n, \frac{1}{a}(K + x) \subset T + y \subset b(K + x)\}.$$

Remark 4.2.7. Given a convex body $K \subset \mathbb{R}^n$ there is a convex body $T \subset \mathbb{R}^n$ such that $\mathrm{vr}(T,K) \sim \mathrm{vr}(K,T) \sim 1$ and $L_T \leqslant c$, where $c > 0$ is an absolute constant.

Indeed, given K, by Theorem 4.2.6 (using $\varepsilon = 1$) there is $T \subset \mathbb{R}^n$ with $L_T \leqslant c$ and $d(K,T) \leqslant 2$. Notice that if for certain $x, y \in \mathbb{R}^n$ and $a, b > 0$ we have that $\frac{1}{a}(K+x) \subset T + y \subset b(K+x)$. Then,

$$\mathrm{vr}(T,K) \leqslant \frac{|T|^{\frac{1}{n}}}{\frac{1}{a}|K|^{\frac{1}{n}}} \leqslant ab\frac{|K|^{\frac{1}{n}}}{|K|^{\frac{1}{n}}} \leqslant ab.$$

Hence $\mathrm{vr}(T,K) \leqslant d(T,K)$, and by symmetry, the same holds for $\mathrm{vr}(K,T)$.

Proposition 4.2.8. *For every convex body $K \subset \mathbb{R}^n$ there is a convex body W with $\mathrm{vr}(W,K) \sim 1$ such that*

$$\|id : \ell_2^n \to X_W\| \sqrt{n} |W|^{\frac{1}{n}} \sim 1. \tag{4.19}$$

Proof of Proposition 4.2.8. By Remark 4.2.7 and the Rogers-Shephard inequality, Theorem 1.2.3, (replacing the body if necessary) we can assume that K° is a centrally symmetric isotropic convex body and L_{K° is uniformly bounded.

Consider W such that $W^\circ = K^\circ \cap c\sqrt{n}B_2^n$, with $c > 0$ the absolute constant in Theorem 4.2.5. This theorem also implies that $|W^\circ|^{\frac{1}{n}} \geqslant (1 - \exp(-\sqrt{n}))^{\frac{1}{n}} \geqslant \frac{1}{2}$ and hence $\mathrm{vr}(W,K) \sim \mathrm{vr}(K^\circ, W^\circ) \sim 1$.

Since $W^\circ \subset c\sqrt{n}B_2^n$ we have that $\|id : \ell_2^n \to X_W\| = \|id : X_{W^\circ} \to \ell_2^n\| \preceq \sqrt{n}$. Finally, as $|W^\circ|^{\frac{1}{n}} \sim 1$, we have that $|W|^{\frac{1}{n}} \sim \frac{1}{n}$ (applying the Blaschke-Santaló/Bourgain-Milman inequality, Theorems 1.2.1 and 1.2.2). Therefore

$$\|id : \ell_2^n \to X_W\| \sqrt{n} |W|^{\frac{1}{n}} \sim 1,$$

which concludes the proof. □

Now we are ready to prove the main result of this chapter. It states that, given K, if we consider any quantity of vectors m, proportional to n, the volume ratio between the random polytope $L^{(m)}$ and K is "large".

Theorem 4.2.9. *Let $K \subset \mathbb{R}^n$ be convex body. Given $\delta \geqslant 1$, with probability greater than or equal to $1 - e^{-n^2}$ the random polytope $L^{(\lceil \delta n \rceil)}$ in (4.4) verifies*

$$\sqrt{n} \preceq \mathrm{vr}(K, L^{(\lceil \delta n \rceil)}).$$

In particular, $\sqrt{n} \preceq \mathrm{lvr}(K)$.

Proof. By Proposition 4.2.8 there is W with $\mathrm{vr}(W,K) \sim 1$ such that

$$\|id : \ell_2^n \to X_W\| \sqrt{n} |W|^{\frac{1}{n}} \sim 1. \tag{4.20}$$

Applying Proposition 4.2.4, given $\delta \geqslant 1$, with probability greater than or equal to $1 - e^{-n^2}$ the random polytope $L^{(\lceil \delta n \rceil)}$ in (4.4) verifies

$$\sqrt{n} \preceq \mathrm{vr}(W, L^{(\lceil \delta n \rceil)}).$$

Then,

$$\sqrt{n} \preceq \mathrm{vr}(W, L^{(\lceil \delta n \rceil)}) \leqslant \mathrm{vr}(W,K)\, \mathrm{vr}(K, L^{(\lceil \delta n \rceil)}) \sim \mathrm{vr}(K, L^{(\lceil \delta n \rceil)}),$$

as wanted. □

The next corollary can be easily derived from the previous theorem by duality.

Corollary 4.2.10. *Let $K \subset \mathbb{R}^n$ be a convex body. Given $\delta \geqslant 1$, there is polytope Z with $2(\lceil \delta n \rceil + n)$ facets such that*

$$\sqrt{n} \preceq \mathrm{vr}(Z^{(\lceil \delta n \rceil)}, K).$$

Proof. By Theorem 4.2.9 there is a polytope L with $2(\lceil \delta n \rceil + n)$ vertices such that $\mathrm{vr}(K^\circ, L) \succeq \sqrt{n}$. Setting $Z := L^\circ$ and applying Proposition 3.1.2 (3) we have that $\mathrm{vr}(Z,K) \sim \mathrm{vr}(K^\circ, L) \preceq \sqrt{n}$. The result follows from the fact that the polar of the polytope $L^{(\lceil \delta n \rceil)}$ has $2(\lceil \delta n \rceil + n)$ facets. □

Chapter 5

Volume ratio between projections of convex bodies

In this chapter we are going to study the volume ratio between projections of two convex bodies. Given $K \subset \mathbb{R}^n$ and $k \sim n$ we show that there is another body Z such that the volume ratio between any projection of rank k of the bodies K and Z is "large". In order to prove the existence of Z we are going to proceed similarly as we did in the previous chapter, using the probabilistic method. Since we need to work with projections of the bodies, we introduce a Gaussian version of the random polytopes. This allow us to easily handle the projections involved. We also use an ε-net argument in order to control the probability for every orthogonal projection of fixed rank.

5.1 Volume ratio of projections

We will now deal with a variation of the problem we have treated in the previous chapter. We denote by $\mathcal{P}^k(n)$ the set of all orthogonal projections of rank k in $\mathbb{R}^n$. For a convex body K we have a collection of k-dimensional convex bodies given by $QK \subset \mathbb{R}^k$ for $Q \in \mathcal{P}^k(n)$. The problem of estimating distances (in the sense of equation (4.6)) between projections of convex bodies had aroused considerable interest (see for example [BS88, Sza90, ST89]).

For a convex body K with 0 as an interior point and a subspace $E \subset \mathbb{R}^n$ one has that $P_E(K^\circ) = (E \cap K)^\circ$, where P_E denotes the orthogonal projection onto E. Hence, every result concerning projections of K has a dual version concerning sections of K°.

It should be mentioned that two convex bodies can be far apart but they may have their projections or sections quite close. That is the case of Gluskin's polytopes defined in the previous chapter. Gluskin proved that, for $m \sim n$, with high probability the distance between two random polytopes

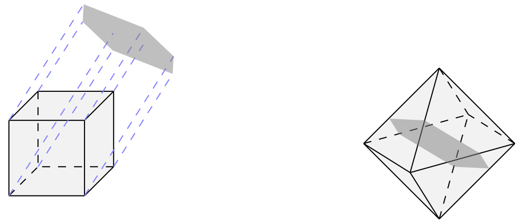

Figure 5.1: A projection of K and a section K°.

of m vertices, say $L^{(m)}$ and $L'^{(m)}$, is greater than n. Despite this, for any k proportional to n, for "most of" of the subspaces $E \subset \mathbb{R}^n$ of dimension k we have that $d_{BM}(L^{(m)} \cap E, L'^{(m)} \cap E) \preceq 1$. In fact, Szarek [Sza79] proved that given a convex body $K \subset \mathbb{R}^n$ with $\mathrm{vr}(K, B_2^n) \sim 1$, and $0 < \delta < 1$, "most of" of the subspaces $F \subset \mathbb{R}^n$ of dimension $k < \delta n$ satisfy that $d_{BM}(K \cap E, B_2^k) \sim 1$. It is easy to verify that for the random polytopes we have $\mathrm{vr}(L^{(m)}, B_2^n) \sim 1$.

Mankiewicz and Tomczak-Jaegermann [MTJ01] found precise estimates of the distance between random k-dimensional sections of two convex bodies in terms of the average distance of a $\frac{k}{2}$-dimensional section of each body to a ball.

In [Rud04], Rudelson studied the problem of estimating extremal distances between sections and projections of convex bodies. For $k < n$, he defined the distance $\delta_k(K, Z)$ as the minimal Banach-Mazur distance between k-dimensional projections of K and Z. He was interested in estimating the diameter of the Banach-Mazur compactum for this "distance". That is, finding the asymptotic behaviour of

$$\Delta(k, n) := \sup \delta_k(K, Z),$$

where the supremum is taken over all n-dimensional convex symmetric bodies K and Z. He proved that

$$\Delta(k, n) \sim_{\log n} \begin{cases} \sqrt{k} & \text{if } k \leqslant n^{2/3} \\ \frac{k^2}{n} & \text{if } k > n^{2/3}, \end{cases}$$

where $A \sim_{log(n)} B$ means that

$$\frac{1}{C \log^a n} A \leqslant B \leqslant (C \log^a n) A$$

for some absolute constants $C, a > 0$. In order to obtain this, he proved that there are convex bodies $K, Z \subset \mathbb{R}^n$, such that for any $k < n$,

$$\delta_k(K, Z) \geq \frac{k^2}{n \log(\log(n))},$$

$$\delta_k(K, B_2^n) \geq \sqrt{\frac{k}{\log(1 + \frac{n}{k})}}.$$

We are going to use Rudelson's approach together with the techniques developed in the previous chapter in order to tackle the volume ratio problem for projections. Namely, given $K \subset \mathbb{R}^n$, we are interested in finding a body $Z \subset \mathbb{R}^n$ such that, for every $Q \in \mathcal{P}^k(n)$, $\mathrm{vr}(QK, QZ)$ is "large". The following theorem is the main result of this chapter.

Theorem 5.1.1. *Given $\delta > 0$ there is a constant $d := d(\delta) > 0$ with the following property:*

for each convex body $K \subset \mathbb{R}^n$ and $\delta n \leqslant k \leqslant n$, there is a centrally symmetric body $Z \subset \mathbb{R}^n$ such that

$$\mathrm{vr}(QK, QZ) \geqslant d\sqrt{\frac{k}{\log\log k}},$$

for every orthogonal projection $Q : \mathbb{R}^n \to \mathbb{R}^n$ of rank k.

Observe that we can also state a dual version of the result.

Corollary 5.1.2. *Given $\delta > 0$ there is a constant $d := d(\delta) > 0$ with the following property:*

for each centrally symmetric convex body $K \subset \mathbb{R}^n$ and $\delta n \leqslant k \leqslant n$, there is a centrally symmetric body $Z \subset \mathbb{R}^n$ such that

$$\mathrm{vr}(E \cap Z, E \cap K) \geqslant d\sqrt{\frac{k}{\log\log k}},$$

for every subspace $E \subset \mathbb{R}^n$ of dimension k.

Proof. Applying Theorem 5.1.1 for K° we have that there is a centrally symmetric body W such that

$$\mathrm{vr}(QK^\circ, QW) \geqslant d\sqrt{\frac{k}{\log\log k}},$$

for every orthogonal projection $Q : \mathbb{R}^n \to \mathbb{R}^n$ of rank k. Given $E \subset \mathbb{R}^n$ of dimension k we have that,

$$\begin{aligned} d\sqrt{\frac{k}{\log\log k}} \leqslant \mathrm{vr}\left(P_E K^\circ, P_E W\right) &\sim \mathrm{vr}\left((P_E W)^\circ), (P_E K^\circ)^\circ\right) \\ &= \mathrm{vr}\left(E \cap W^\circ, E \cap K\right), \end{aligned}$$

since, $P_E K^\circ = E \cap K$ and $P_E W = E \cap W^\circ$ the result follows by taking $Z = W^\circ$. □

Before starting with the proof we recall a standard result in geometric measure theory,see e.g., [Mat99, Theorem 7.5].

Theorem 5.1.3. *If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz map, $0 \leqslant s \leqslant m$, and $A \subset \mathbb{R}^m$, then*

$$\mathcal{H}^s(f(A)) \leqslant Lip(f)^s \mathcal{H}^s(A),$$

where $\mathcal{H}^s$ and $Lip(f)$ are the s-Hausdorff measure and the Lipschitz constant of f, respectively.

Recall that, for $k \in \mathbb{N}$, the k-Hausdorff measure is a multiple of the Lebesgue measure in $\mathbb{R}^k$. Namely, for every measurable set $\mathcal{H}^k(A) = \frac{2^k}{|B_2^k|}|A|$.

Given $Q \in \mathcal{P}^k(n)$ we will denote by $|QK|$ the k-dimensional Lebesgue measure of QK. As an application of the last theorem we have the following lemma, that relates the k-dimensional volume of two different projections of K with their distance in the canonical operator metric.

Lemma 5.1.4. *Let $P, Q \in \mathcal{P}^k(n)$ such that $\|P - Q\| \leqslant \frac{1}{\sqrt{n}}$. For every centrally symmetric convex body $K \subset \mathbb{R}^n$ in John's position,*

$$\frac{1}{3}|QK|^{\frac{1}{k}} \leqslant |PK|^{\frac{1}{k}} \leqslant 3|QK|^{\frac{1}{k}}.$$

Proof. First we will see that

$$PK \subset 3PQK. \tag{5.1}$$

Take $x \in K$,

$$\begin{aligned} Px &= PQx + P(x - Qx) \\ &= PQx + P(x - Px) + P(Px - Qx) \\ &= PQx + P(P - Q)x. \end{aligned}$$

Thus, to prove (5.1) it is sufficient to show that

$$P(P - Q)x \in 2PQK. \tag{5.2}$$

It is easy to see that the operator P coincides with $I + P - Q$ on $Q\mathbb{R}^n$ (the image of the projection Q). Since $\|P - Q\| \leqslant \frac{1}{\sqrt{n}}$, then

$$P|_{Q\mathbb{R}^n} : Q\mathbb{R}^n \to P\mathbb{R}^n$$

is invertible and its inverse $S := (P|_{Q\mathbb{R}^n})^{-1} = \sum_{k=0}^{\infty}(Q - P)^k$ satisfies

$$\|S\| \leqslant \frac{1}{1 - \frac{1}{\sqrt{n}}} \leqslant 2.$$

Note that $PB_2^n \subset 2PQB_2^n$. Indeed, by the previous estimate about the norm of S we have $SPB_2^n \subset 2QB_2^n$ and so, applying P, we get $PSPB_2^n = PB_2^n \subset 2PQB_2$.

Since $K \subset \sqrt{n}B_2^n$ (K is in John's position) we get that $(P-Q)x \subset B_2^n$ and

$$P(P-Q)x \in PB_2^n \subset 2PQB_2^n \subset 2PQK.$$

This shows (5.2), which as we mentioned implies (5.1).

To finish the proof, with equation (5.1) at hand, we just apply Theorem 5.1.3 with $m := n$, $s := k$, $f := P$ and $A := QK$ to obtain

$$|PK|^{\frac{1}{k}} \leqslant 3|PQK|^{\frac{1}{k}} \leqslant 3|QK|^{\frac{1}{k}},$$

using that the Lipschitz constant of the mapping P is obviously one and simplifying the constants to pass from Hausdorff to Lebesgue measure. □

5.2 Gaussian polytopes

We now define a variant of the Gluskin's random polytopes defined in Section 4.2. Instead of considering the absolute convex hull of points taken uniformly on the unit sphere we are going to work with Gaussian random vectors. The reason for doing this is that we want to deal with projections of these bodies, and the Gaussian measure is more suitable for this purpose. Let $N > n$ and $g_1, \ldots, g_N$ be standard independent Gaussian vectors in $\mathbb{R}^n$. We define the symmetric polytope

$$Z_N = Z_N(\omega) = \text{absconv}\{\sqrt{n}e_1, \ldots, \sqrt{n}e_n, g_1, \ldots, g_N\}.$$

We are going to need the following lemma regarding the euclidean norm of a Gaussian vector.

Lemma 5.2.1. *Let g be a standard Gaussian vector. Then, there are constants $C, c > 0$ such that $1 \leqslant \|g\|_2 \leqslant C\sqrt{n}$ with probability at least $1 - e^{-cn}$.*

Proof. Bounding the gaussian density by $(2\pi)^{-\frac{n}{2}}$. We have

$$\mathbb{P}\{\|g\|_2 < 1\} \leqslant \frac{|B_2^n|}{(2\pi)^{\frac{n}{2}}} = \frac{\left(2\Gamma(1+\frac{1}{2})\right)^n}{(2\pi)^{\frac{n}{2}}\Gamma(1+\frac{n}{2})},$$

applying Stirling's formula and choosing an appropriate constant $c_1 > 0$ we have,

$$\mathbb{P}\{\|g\|_2 \geqslant 1\} \geqslant 1 - e^{-c_1 n}. \tag{5.3}$$

Now, if $g = (g_1, \ldots, g_n)$ we want to bound,

$$\mathbb{P}\left\{\left(\sum_{i=1}^n g_i^2\right)^{1/2} \geqslant C\sqrt{n}\right\} = \mathbb{P}\left\{e^{\frac{1}{4}\left(\sum_{i=1}^n g_i^2\right)} \geqslant e^{C\frac{1}{4}n}\right\}.$$

Appyling Markov's inequality,

$$\begin{aligned} P\left\{e^{\frac{1}{4}(\sum_{i=1}^n g_i^2)} \geqslant e^{\frac{1}{4}Cn}\right\} &\leqslant e^{-\frac{1}{4}Cn}(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{\frac{\|x\|^2}{4}}e^{-\frac{\|x\|^2}{2}}dx \\ &= e^{-\frac{1}{4}Cn}\left((2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}} e^{-\frac{1}{4}t^2}dt\right)^n. \end{aligned}$$

If we compute the right hand integral and choose an appropriate constant $c_2 > 0$ we have,

$$\mathbb{P}\left\{\left(\sum_{i=1}^n g_i^2\right)^{1/2} \geqslant C\sqrt{n}\right\} \leqslant 1 - e^{-c_2 n}. \tag{5.4}$$

Finally we use the union bound to combine (5.3) and (5.4). □

A consequence of the last lemma is that the set

$$\Omega_0 := \{\omega | B_2^n \subset Z_N(\omega) \subset C\sqrt{n}B_2^n\}$$

satisfies

$$\mathbb{P}(\Omega_0) \geqslant 1 - Ne^{-cn}. \tag{5.5}$$

To estimate the volume of projections of Z_N recall that, by Lemma 3.2.1, if $v_1, \ldots, v_N$ are vectors in $\mathbb{R}^m$ of lenght at most one then,

$$|\operatorname{absconv}\{v_1, \ldots, v_N\}|^{1/m} \leqslant c\frac{\sqrt{\log(1 + N/m)}}{m}.$$

Thus, for each $w \in \Omega_0$ and $Q \in \mathcal{P}^k(n)$ we have

$$|QZ_N(w)|^{1/k} \leqslant C\frac{\sqrt{n\log(1 + N/k)}}{k}. \tag{5.6}$$

We will now prove a series of lemmas that are similar to the ones that we stated in Section 4.1. Given a finite dimensional normed space X, we denote by $\mathcal{S}(X)$ the set of all linear operators $T : X \to X$ of determinant one.

Lemma 5.2.2. *Let $K \subset \mathbb{R}^n$ a centrally symmetric convex body, $Q_0 \in \mathcal{P}^k(n)$ and $T_0 \in \mathcal{S}(Q_0\mathbb{R}^n)$ fixed. For $A > 0$ we have*

$$\mathbb{P}\{\omega \in \Omega_0 : \|T_0 : X_{Q_0 Z_N(\omega)} \to X_{Q_0 K}\| \leqslant A\} \leqslant C^{kN}A^{kN}|Q_0 K|^N.$$

Proof. Observe that

$$\begin{aligned} \{\omega \in \Omega_0 : T_0 Q_0(Z_N(\omega)) \subset AQ_0 K\} &= \{\omega \in \Omega_0 : Q_0 Z_N(\omega) \subset AT_0^{-1}(Q_0 K)\} \\ &\subset \{\omega \in \Omega_0 : Q_0 g_i(\omega) \in AT_0^{-1}(Q_0 K) \text{ for all } 1 \leqslant i \leqslant N\}. \end{aligned}$$

Using the rotational invariance of the measure and the fact that T_0 preserves measure (it has determinant one), we have

$$\begin{aligned}
&\mathbb{P}\{\omega \in \Omega_0 : \|T_0 : X_{Q_0 Z_N(\omega)} \to X_{Q_0 K}\| \leqslant A\} \\
&\qquad \leqslant \mathbb{P}\{\omega \in \Omega_0 : Q_0 g_1(\omega) \in A T_0^{-1} Q_0 K\}^N \\
&\qquad = \left(\int_{\mathbb{R}^n} \mathbb{1}_{A T_0^{-1} Q_0 K}(Q_0 x) d\gamma_k(x) \right)^N \\
&\qquad = \left(\int_{\mathbb{R}^k} \mathbb{1}_{A T_0^{-1} Q_0 K}(y) d\gamma_n(y) \right)^N \\
&\qquad \leqslant \frac{1}{(\sqrt{2\pi})^{kN}} (|A T_0^{-1} Q_0 K|^k)^N \\
&\qquad \leqslant C^{kN} A^{kN} |Q_0 K|^N.
\end{aligned}$$

This concludes the proof. □

Lemma 5.2.3. *Let $K \subset \mathbb{R}^n$ be a centrally symmetric convex body, $Q_0 \in \mathcal{P}^k(n)$ a fixed orthogonal projection of rank k and $A > 0$, then*

$$\begin{aligned}
&\mathbb{P}\{\omega \in \Omega_0 : \exists T \in \mathcal{S}(Q_0 \mathbb{R}^n) \text{ and } \|T : X_{Q_0 Z_N(\omega)} \to X_{Q_0 K}\| \leqslant A\} \\
&\qquad \leqslant (c\sqrt{n})^{k^2} C^{Nk} A^{Nk} |Q_0 K|^N.
\end{aligned}$$

Proof. Let $U := B_{\mathcal{L}(\ell_2^k, X_{Q_0 K})}$ the unit ball of $\mathcal{L}(\ell_2^k, X_{Q_0 K})$, and consider $\mathcal{N}$ a maximal set $\frac{A}{\sqrt{n}}$- separated in $AU \cap SL(k, R)$ for the metric $\| \; \|_{\mathcal{L}(\ell_2^k, X_{Q_0 K})}$.

Therefore we have the following inclusion for the disjoint union

$$\bigcup_{\eta \in \mathcal{N}} (\eta + \frac{A}{2\sqrt{n}} U) \subset (1 + \frac{1}{2\sqrt{n}}) AU.$$

Identifying the space with $\mathbb{R}^{k^2}$ and taking measure we conclude that

$$\# \mathcal{N} \leqslant (c\sqrt{n})^{k^2}.$$

Take $\omega \in \Omega_0$ such that there is $T \in \mathcal{S}(Q_o \mathbb{R}^n)$ with $\|T : X_{Q_0 Z_N(\omega)} \to X_{Q_0 K}\| \leqslant A$. Since $B_2^n \subset Z_N(\omega)$ we have that $T \in AU$, so there is $S \in \mathcal{N}$ such that

$$\|S - T : \ell_2^k \to X_{Q_0 K}\| \leqslant \frac{A}{\sqrt{n}}.$$

Then,

$$\begin{aligned}
\|S : X_{Q_0 Z_N(\omega)} \to X_{Q_0 K}\| &\leqslant \|S - T : X_{Q_0 Z_N(\omega)} \to X_{Q_0 K}\| + \|T : X_{Q_0 Z_N(\omega)} \to X_{Q_0 K}\| \\
&\leqslant \sqrt{n} \|S - T : \ell_2^k \to X_{Q_0 K}\| + A \\
&\leqslant 2A.
\end{aligned}$$

This shows

$$\{\omega \in \Omega_0 : \exists T \in \mathcal{S}(Q_0\mathbb{R}^n) \text{ and } \|T : X_{Q_0 Z_N(\omega)} \to X_{Q_0 K}\| \leqslant A\}$$
$$\subset \bigcup_{S \in \mathcal{N}} \{S : \|S : \ell_2^k \to X_{Q_0 K}\| \leqslant 2A\}.$$

Taking measure and applying Lemma 5.2.2 we obtain the desired bound. □

We will need the following result about ε-nets due to Szarek [Sza82].

Lemma 5.2.4. *Let $0 < \varepsilon < 1$. The set $\mathcal{P}^k(n)$ admits an ε-net Π of cardinality*

$$|\Pi| \leqslant \left(\frac{C}{\varepsilon}\right)^{nk}.$$

Given a basis $B = \{v_1, \dots, v_k\}$ of a vector space F and a vector $x \in F$ we denote by $(x)_B$ the coordinates of x in the basis B. That is, $(x)_B = (\alpha_1, \dots, \alpha_k)$ if $x = \sum_{i=1}^k \alpha_i v_i$. Also for an operator $T : F \to F$ we denote by $[T]_B$ the matrix $(a_{i,j})_{1 \leqslant i,j \leqslant k}$ such that $T(v_l) = \sum_{i=1}^k a_{i,l} v_i$, for every $1 \leqslant l \leqslant k$ (i.e., the l-column of $[T]_B$ is $(Tv_l)_B^t$).

Lemma 5.2.5. *Given a centrally symmetric convex body $K \subset \mathbb{R}^n$ and $\beta > 0$ we have*

$$\mathbb{P}\{\omega \in \Omega_0 | \ \exists Q \in \mathcal{P}^k(n), \exists T \in \mathcal{S}(Q\mathbb{R}^n) \ \textit{such that} \ \|T : X_{QZ_N(\omega)} \to X_{QK}\| \leqslant \frac{\beta}{|QK|^{\frac{1}{k}}}\}$$
$$\leqslant (c_1\sqrt{n})^{nk} (c_2\sqrt{n})^{k^2} c_3^{kN} \beta^{kN}.$$

Proof. Assume that K is in John's position. By the previous lemma there is a $\frac{1}{\sqrt{n}}$-net for $\mathcal{P}^k(n)$ of cardinality $\#\Pi \leqslant (c_1\sqrt{n})^{nk}$. In order to prove the result we need to show that

$$\left\{\omega \in \Omega_0 | \exists Q \in \mathcal{P}^k(n), \exists T \in \mathcal{S}(Q\mathbb{R}^n) \text{ such that } \|T : X_{QZ_N(\omega)} \to X_{QK}\| \leqslant \frac{\beta}{|QK|^{\frac{1}{k}}}\right\}$$
$$\subset \bigcup_{Q_0 \in \Pi} \left\{\omega \in \Omega_0 | \exists S \in \mathcal{S}(Q_0\mathbb{R}^n) \text{ such that } \|S : X_{Q_0 Z_N(\omega)} \to X_{Q_0 K}\| \leqslant C\frac{\beta}{|Q_0 K|^{\frac{1}{k}}}\right\}.$$

The bound follows by applying Lemma 5.2.3.

Let $\omega \in \Omega_0$ such that there is $Q \in \mathcal{P}^k(n)$ and $T \in \mathcal{S}(Q\mathbb{R}^n)$ with

$$\|T : X_{QZ_N(\omega)} \to X_{QK}\| \leqslant \frac{\beta}{|QK|^{\frac{1}{k}}}.$$

Take $Q_0 \in \Pi$ such that $\|Q - Q_0\| \leqslant \frac{1}{\sqrt{n}}$. Fix and orthonormal basis $B = \{v_1, \dots, v_k\}$ of $Q\mathbb{R}^n$. It is easy to see that the collection $\tilde{B} = \{Q_0 v_1, \dots, Q_0 v_k\}$ is a basis of $Q_0\mathbb{R}^n$. Define S such that $[S]_{\tilde{B}} = [T]_B$, so $S \in \mathcal{S}(Q_0\mathbb{R}^n)$.

We now show that

$$\|S : X_{Q_0 Z_N(\omega)} \to X_{Q_0 K}\| \leqslant C \frac{\beta}{|QK|^{\frac{1}{k}}}. \tag{5.7}$$

Indeed, take $x \in Z_N(\omega)$,

$$SQ_0 x = \underbrace{SQ_0(Q_0 x - Qx)}_{(1)} + \underbrace{SQ_0 Qx}_{(2)}.$$

We must see that the terms (1) and (2) belong to $C\frac{\beta}{|QK|^{\frac{1}{k}}} Q_0 K$.

To see that the term (2), SQ_0Qx, is in $C\frac{\beta}{|QK|^{\frac{1}{k}}} Q_0 K$, write $Qx = \sum \alpha_i v_i$, so $Q_0 Qx = \sum \alpha_i Q_0 v_i$. We have,

$$\begin{aligned}(SQ_0Qx)^t_{\tilde{B}} &= [T]_B (Q_0 Qx)^t_{\tilde{B}} \\ &= [T]_B (Qx)^t_B \\ &= (TQx)^t_B.\end{aligned}$$

Hence, $SQ_0Qx = Q_0TQx$. Since $x \in Z_N(\omega)$, $TQx \in \frac{\beta}{|QK|^{\frac{1}{k}}} QK$ and then $Q_0TQx \in \frac{\beta}{|QK|^{\frac{1}{k}}} Q_0 QK$. Now notice that if we assume that K is in John's position (hence $B_2^n \subset K \subset \sqrt{n} B_2^n$), we have that $Q_0 QK \subset 2 Q_0 K$. This is because

$$\begin{aligned}Q_0 QK \subset& Q_0 K + Q_0((Q - Q_0)K) \\ \subset& Q_0 K + Q_0((Q - Q_0)\sqrt{n} B_2^n) \\ \subset& Q_0 K + Q_0 K \\ =& 2 Q_0 K.\end{aligned}$$

Now we are going to prove that the term (1), $SQ_0(Q_0x - Qx)$, is in $C\frac{\beta}{|QK|^{\frac{1}{k}}} Q_0 K$. Since $Z_N(\omega) \subset \sqrt{n} B_2^n$, $(Q_0 - Q)x \in B_2^n$ and then $Q_0(Q_0 - Q)x \in B_2^k$. We need to see that $\|S : \ell_2^k \to X_{Q_0 K}\| \leqslant C\frac{\beta}{|QK|^{\frac{1}{k}}}$.

Take $y \in Q_0 \mathbb{R}^n$ with $\|y\|_2 = 1$. We write $(y)_{\tilde{B}} = (\beta_1, \dots, \beta_k)$ and $(Sy)_{\tilde{B}} = (\gamma_1, \dots, \gamma_k)$. Then,

$$\begin{aligned}(\gamma_1, \dots, \gamma_k) =: [Sy]_{\tilde{B}} =& [T]_B (\beta_1, \dots, \beta_k)^t \\ =& [T(\sum \beta_i v_i)]_B.\end{aligned}$$

Notice that

$$\begin{aligned}\|\sum \beta_i v_i\|_2 \leqslant& \|\sum \beta_i Q v_i - \sum \beta_i Q_0 v_i\|_2 + \|\sum \beta_i Q_0 v_i\|_2 \\ \leqslant& \frac{1}{\sqrt{n}} \|\sum \beta_i v_i\|_2 + 1,\end{aligned}$$

so,

$$\left(1 - \frac{1}{\sqrt{n}}\right) \| \sum \beta_i v_i \|_2 \leqslant 1$$
$$\| \sum \beta_i v_i \|_2 \leqslant \frac{1}{1 - \frac{1}{\sqrt{n}}} \leqslant 2.$$

Then, $T(\sum \beta_i v_i) \in \frac{2\beta}{|QK|^{\frac{1}{k}}} QK$. On the other hand,

$$Sy = \sum \gamma_i Q_0 v_i = Q_0(\sum \gamma_i v_i) = Q_0 T(\sum \beta_i v_i).$$

So we have that $Sy \in \frac{2\beta}{|QK|^{\frac{1}{k}}} Q_0 QK \subset \frac{4\beta}{|QK|^{\frac{1}{k}}} Q_0 K$. The result now follows because $|QK|^{\frac{1}{k}} \sim |Q_0 K|^{\frac{1}{k}}$, using Lemma 5.1.4.

□

We are now ready to prove our main result.

Proof. (of Theorem 5.1.1) By the Roger-Shephard inequality we know that $\mathrm{vr}(L - L, L) \leqslant 4$, for every convex body $L \subset \mathbb{R}^n$. Therefore, since $Q(K - K) = Q(K) - Q(K)$, for every $Q \in \mathcal{P}^k(n)$ we have that

$$\mathrm{vr}(Q(K - K), Z) \leqslant \mathrm{vr}(Q(K - K), QK)\, \mathrm{vr}(QK, Z) \leqslant 4\, \mathrm{vr}(QK, Z).$$

So, if $\mathrm{vr}(Q(K - K), Z)$ is large so is $\mathrm{vr}(QK, Z)$. So we can suppose without loss of generality, from now on that K is centrally symmetric.

By Lemma 5.2.5 we know that

$$\mathbb{P}\{\omega \in \Omega_0 | \exists Q \in \mathcal{P}^k(n), \exists T \in \mathcal{S}(Q\mathbb{R}^n)\ \|T : X_{QZ_N(\omega)} \to X_{QK}\| \leqslant \frac{\beta}{|QK|^{\frac{1}{k}}}\}$$
$$\leqslant (c_1\sqrt{n})^{nk} (c_2\sqrt{n})^{k^2} c_3^{kN} \beta^{kN}. \tag{5.8}$$

Let $N = n \log(n)$ and fix β small enough so that the probability in (5.8) tends to zero. Hence, since $\mathbb{P}(\Omega_0) \geqslant 1 - Ne^{-cn}$ (Equation (5.5)), we know there is $\omega \in \Omega_0$ such that for every $Q \in \mathcal{P}^k(n)$ and $T \in \mathcal{S}(Q\mathbb{R}^n)$,

$$\|T : X_{QZ_N(\omega)} \to X_{QK}\| \geqslant \frac{\beta}{|QK|^{\frac{1}{k}}}.$$

If we compute the volume ratio between projections of $Z := Z_N(\omega)$ and K we get, using that $\delta n \leqslant k \leqslant n$, Proposition 3.1.2 (1) and equation (5.6),

$$\begin{aligned}
\mathrm{vr}(QK, QZ) \geqslant & \frac{\beta}{|QK|^{\frac{1}{k}}} \frac{|QK|^{\frac{1}{k}}}{|QZ|^{\frac{1}{k}}} \\
\geqslant & \frac{\beta k}{C\sqrt{n \log(2 + N/k)}} \\
\geqslant & d(\delta) \frac{\sqrt{k}}{\sqrt{\log\log(k)}},
\end{aligned}$$

which concludes the proof of the main theorem. □

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