

The Minimal Volume of Simplices Containing a Convex Body

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Abstract Let $K \subset \mathbb{R}^n$ be a convex body with barycenter at the origin. We show there is a simplex $S \subset K$ having also *barycenter at the origin* such that $(\frac{\text{vol}(S)}{\text{vol}(K)})^{1/n} \geq \frac{c}{\sqrt{n}}$, where $c > 0$ is an absolute constant. This is achieved using stochastic geometric techniques. Precisely, if K is in isotropic position, we present a method to find centered simplices verifying the above bound that works with extremely high probability. By duality, given a convex body $K \subset \mathbb{R}^n$ we show there is a simplex S enclosing K with the same barycenter such that

$$\left(\frac{\text{vol}(S)}{\text{vol}(K)}\right)^{1/n} \leq d\sqrt{n},$$

for some absolute constant $d > 0$. Up to the constant, the estimate cannot be lessened.

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1 Introduction

Approximating a geometric body by a much simpler one results a very common technique in convex geometry and convex analysis, with many applications in discrete geometry and discrete/continuous optimization. For example, the use of the John/Löwner ellipsoid (maximum/minimum volume ellipsoid, respectively) is one of the most standard tools in these areas [13, 15, 22, 23, 25, 30]. Polytopes, next to ellipsoids, are the most elementary convex sets, chief among them is the simplex. Extremal convex sets for volume ratios of the Euclidean ball are exactly the simplices [3, 4].

Throughout this article, simplices are n -simplices in \mathbb{R}^n exclusively, i.e., those polytopes formed by the convex hull of $(n + 1)$ affine independent points in \mathbb{R}^n (the vertices). A convex body in \mathbb{R}^n is a compact convex set with non-empty interior. For a bounded subset $A \subset \mathbb{R}^n$, we denote by $\text{vol}(A)$ the volume (or Lebesgue measure) of A .

Given a convex body $K \subset \mathbb{R}^n$, we define

$$S(K) := \min \left(\frac{\text{vol}(S)}{\text{vol}(K)} \right)^{1/n},$$

where the minimum is taken over all simplices S in \mathbb{R}^n containing K . An old problem in convex geometry is the following:

Problem 1.1 *How large can $S(K)$ be?*

For the Euclidean plane, i.e., $n = 2$, this problem was completely solved by Gross [14] (and generalized in different ways by W. Kuperberg [21]): every convex body $K \subset \mathbb{R}^2$ can be inscribed in a triangle of area at most $2\text{vol}(K)$. This ratio corresponds (exclusively) to the case that K is a parallelogram. The measure of the tetrahedron (not necessarily regular) of least volume circumscribed around a convex body $K \subset \mathbb{R}^3$ is unknown. If $K \subset \mathbb{R}^3$ is a parallelepiped of volume one, then the minimal volume tetrahedron containing it has volume $9/2$. It is an open question whether this is the worst possible fit for the general case. To our knowledge, there are not even conjectured bounds for greater dimensions ($n \geq 4$).

Asymptotic results on this problem were given in the seventies by Chakerian [9, Corollary 5]. The same estimate was recently rediscovered in 2014 by Kanazawa [17, Theorem 1] using different arguments. In particular, both authors showed that

$$S(K) \leq n^{\frac{n-1}{n}} \approx n. \quad (1)$$

Note that when $n = 2$ this is just Gross' bound.

It is possible to improve the previous bound applying a general inequality for volume ratios due to Giannopoulos and Hartzoulaki [12]. As a consequence of their results we have

$$S(K) \leq c\sqrt{n} \log(n), \tag{2}$$

where $c > 0$ is an absolute constant. Up to our knowledge, this was the best known bound; see also the bound given in the recent work of Paouris and Pivovarov [32, Corollary 5.4] on this problem.

One might be interested in requiring additional properties to the simplex. For example, that it shares the same barycenter as the given convex body. This induces a strong version of the aforementioned problem. Given a convex body $K \subset \mathbb{R}^n$, we define

$$S_{\circ}(K) := \min \left(\frac{\text{vol}(S)}{\text{vol}(K)} \right)^{1/n},$$

where the minimum is taken over all simplices S containing K and having the same barycenter. Recall that the barycenter (or center of mass) of a convex body K is given by

$$\text{bar}(K) := \frac{1}{\text{vol}(K)} \int_K x dx. \tag{3}$$

Problem 1.2 *How large can $S_{\circ}(K)$ be?*

Our main result is the following asymptotic estimate on this problem.

Theorem 1.3 *Let $K \subset \mathbb{R}^n$ be a convex body. There is a simplex S enclosing K with the same barycenter such that*

$$\left(\frac{\text{vol}(S)}{\text{vol}(K)} \right)^{1/n} \leq d\sqrt{n}, \tag{4}$$

for some absolute constant $d > 0$.

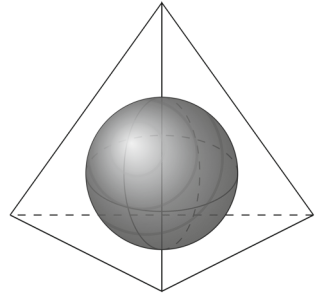
In fact for a centrally symmetric convex body K we prove the following bound:

$$S_{\circ}(K) \leq \frac{d\sqrt{n}}{L_{K^{\circ}}}, \tag{5}$$

for some absolute constant $d > 0$. Here $L_{K^{\circ}}$ stands for the isotropic constant of the polar body K° (see definitions below). If K is an arbitrary body (not necessarily centrally symmetric), we have

$$S_{\circ}(K) \leq \frac{d\sqrt{n}}{L_{D(K)^{\circ}}}, \tag{6}$$

Fig. 1 The simplex of minimal volume enclosing the Euclidean ball is the regular simplex circumscribing it



where $D(K)$ stands for the difference body $K - K$. Note that Eq. (5) is a direct consequence of Eq. (6) (if K is centrally symmetric then the difference body $D(K)$ is just $2 \cdot K$).

The estimate (4), up to the absolute constant $d > 0$, cannot be improved. Indeed, we show in Example 2.7 that if $K = B_2^n$, the Euclidean unit ball, then the regular simplex circumscribing it (see Fig. 1) is a minimal volume simplex that contains K , and therefore

$$S(K) = S_o(K) \geq \tilde{d}\sqrt{n}, \quad (7)$$

for some positive constant $\tilde{d} > 0$.

By duality, Problem 1.2 is related with finding simplices of large volume *inside* a convex body with the same barycenter. The search of simplices of large volume contained in a convex body has an extensive and interesting history in geometry. For instance, the study of the maximum area of triangles in planar convex bodies was undertaken by Blaschke [6] in the early 20th century. Sas [34] and Macbeath [24] also considered the problem of approximating a given convex body by inscribed polytopes. McKinney [26] studied certain properties of those simplices of maximum volume inside a centrally symmetric convex body. The survey [16] also deals with simplices of large volume in cubes.

Given a $K \subset \mathbb{R}^n$ be a convex body with barycenter at the origin, we focus on finding a simplex $S \subset \mathbb{R}^n$ of large volume having also *barycenter at the origin*. Our contribution is the following.

Theorem 1.4 *Let $K \subset \mathbb{R}^n$ be a convex body with barycenter at the origin. There is a simplex $S \subset K$ with barycenter at the origin such that*

$$\left(\frac{\text{vol}(S)}{\text{vol}(K)} \right)^{1/n} \geq \frac{c}{\sqrt{n}}, \quad (8)$$

where $c > 0$ is an absolute constant.

Recall that the Mahler product of a given convex set $K \subset \mathbb{R}^n$ is defined as

$$M(K) := \text{vol}(K)\text{vol}(K^\circ), \quad (9)$$

where K° stands for the polar set of K , i.e.,

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K\}. \tag{10}$$

One of the reasons we restrict ourselves in searching for simplices having *barycenter at origin* is because we know the exact value of their Mahler product (see Lemma 2.6).

Our approach to obtain Theorem 1.3 is based on a very simple idea. Loosely speaking, it is not difficult to see that the problem can be reduced to the case in which K is centrally symmetric (and therefore K° has barycenter at the origin). Note that, by Theorem 1.4, there is a simplex $T \subset K^\circ$ of *large* volume having also barycenter at the origin. By duality, we have that K is enclosed by the simplex T° (with barycenter at the origin), which we show that has small volume. To do this, we make use of its Mahler product (since T is centered) and the reverse Santaló inequality (also known as the Bourgain–Milman inequality, [1, Theorem 8.2.2]) for the body K .

Stochastic geometry studies randomly generated geometric objects. We use techniques from this area to prove Theorem 1.4. Indeed, this theorem is a consequence of a more general result of probabilistic nature (see Theorem 1.5).

Before we go into more detail, we recall some basic definitions and set some notation. We denote the family of all simplices in \mathbb{R}^n with barycenter at the origin by \mathcal{S}_0^n . We write S^{n-1} for the Euclidean sphere in \mathbb{R}^n and denote by $\langle \cdot, \cdot \rangle$ the standard scalar product in \mathbb{R}^n .

A convex body is said to be in isotropic position (or simply, is isotropic) if it has volume one and satisfies the following two conditions:

- $\int_K x \, dx = 0$ (barycenter at 0),
- $\int_K \langle x, \theta \rangle^2 \, dx = L_K^2 \quad \forall \theta \in S^{n-1}$,

where L_K is a constant independent of θ , which is called the isotropic constant of K .

It is not hard to see that for every convex body K in \mathbb{R}^n with center of mass at the origin, there exists $A \in GL(n)$ such that $A(K)$ is isotropic [1, Proposition 10.1.3]. Moreover, this isotropic image is unique up to orthogonal transformations; consequently, the isotropic constant L_K results in an invariant of the linear class of K . In some sense, the isotropic constant L_K measures the spread of a convex body K .

If the convex body K is in isotropic position, the following theorem gives a probabilistic method to find simplices inside K (having barycenter at the origin) with volume large enough. We believe this result is interesting in its own right.

Theorem 1.5 *There exists a function $f_n : \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_n \rightarrow \mathcal{S}_0^n$ such that for every isotropic convex body $K \subset \mathbb{R}^n$ and X_1, \dots, X_n independent random vectors uniformly distributed on K , then with probability greater than $1 - e^{-n}$ we have that $f_n(X_1, \dots, X_n)$ is a simplex with barycenter at the origin contained in K such that*

$$\text{vol}(f_n(X_1, \dots, X_n)) \geq \frac{c^n L_K^n}{n^{n/2}}, \tag{11}$$

where $c > 0$ is an absolute constant.

Note that the volume ratio is invariant under linear transformations, i.e.,

$$\left(\frac{\text{vol}(S)}{\text{vol}(K)}\right)^{1/n} = \left(\frac{\text{vol}(A(S))}{\text{vol}(A(K))}\right)^{1/n} \quad (12)$$

for every $A \in GL(n)$. Thus, we have the following result:

For every convex body $K \subset \mathbb{R}^n$ with barycenter at the origin, there is a simplex $S \subset K$ having also barycenter at the origin such that

$$\left(\frac{\text{vol}(S)}{\text{vol}(K)}\right)^{1/n} \geq \frac{cL_K}{\sqrt{n}}, \quad (13)$$

where $c > 0$ is an absolute constant.

Observe that, since the isotropic constant L_K of any convex body is bounded by below by an absolute constant [1, Proposition 10.1.8], then Theorem 1.4 follows from the previous inequality. We emphasize that it is unknown whether the isotropic constant is bounded above by an absolute constant. The best known general upper bound is $L_K \leq cn^{\frac{1}{4}}$, which was given by Klartag [19] and improves the earlier estimate $L_K \leq cn^{\frac{1}{4}} \log n$ due to Bourgain [7].

Estimates (11) and (13) should also be contrasted with a classic result of Macbeath [24] (see also [29, Theorem 2.10]), which asserts that any convex body $K \subset \mathbb{R}^n$ contains a convex polytope of d vertices, whose volume is at least as large as the maximal volume of a polytope of d vertices inscribed in a Euclidean ball (of the same volume as K). In particular, if $d = n + 1$ it is not difficult to see that we can find a simplex $S \subset K$ such that

$$\left(\frac{\text{vol}(S)}{\text{vol}(K)}\right)^{1/n} \geq \frac{1}{\sqrt{n}}, \quad (14)$$

where $c > 0$ is an absolute constant. The same can be deduced using the well-known Dvoretzky theorem. The inequalities given in (11), (13), and (14) resemble, at first glance, the asymptotic growth given by Milman and Pajor in [27, Proposition 5.6] (connected with the n -dimensional generalization of the classical Sylvester problem, see [5] and the references therein). On the other hand, Eqs. (11) and (13) can also be linked with the bounds given when applying the Blaschke–Groemer inequality [35, Theorem 8.6.3] (or the Busemann random simplex inequality, see [11, Theorem 9.2.6] or [35, Theorem 8.6.1]), which states that the expected volume of a randomly generated simplex inside a given convex body K of fixed volume is minimized when K is an ellipsoid. Anyway, either by the results of Milman and Pajor [27, Proposition 5.6], or by the Blaschke–Groemer inequality [35, Theorem 8.6.3], for every convex body $K \subset \mathbb{R}^n$ one gets

$$\mathbb{E}_{X_i \in K} [\text{vol}(\text{conv}(X_1, \dots, X_{n+1}))] \geq \frac{c^n \text{vol}(K)^n}{n^{n/2}}, \quad (15)$$

where $c > 0$ is an absolute constant.

Our contribution, Theorem 1.5, consists in giving *with extremely high probability*, simplices with the **same barycenter** (a key property for our purposes) whose volumes satisfy the same lower bound of order $\frac{\text{vol}(K)^n}{n^{n/2}}$. The main idea to get this is to show that we can find with extremely high probability randomly generated simplices whose *barycenters are close to the origin* (Proposition 2.1) and *with large volume* (see Proposition 2.4; this should also be compared with Eq. (15)). Then we make a suitable rescale to make the centroids match, with the care to keep staying within the original body. All this is inspired, in a sense, on some arguments presented on the recent paper of Naszódi [28], which solves a conjecture of Bárány, Katchalski, and Pach regarding quantitative Helly-type results (see also the proof of [8, Theorem 3.1]).

The article is organized as follows. In Sect. 2 we give a proof of Theorem 1.5. Finally, in Sect. 2.1 we prove Theorem 1.3, and exhibit that the corresponding asymptotic estimate is sharp. We refer the reader to the books [1,5] for the general theory of asymptotic geometric analysis and the theory of isotropic convex bodies. In Sect. 3 we have included an alternative proof of Theorem 1.3 based on some enlightening comments given by the anonymous referee.

2 A Probabilistic Approach

The probabilistic method is a standard method for proving the existence of a specified kind of mathematical object. The philosophy is to show that if one randomly chooses objects from a specified class, the probability that the result is of the prescribed type is positive. In this section, we use this method to give a proof Theorem 1.5. For this we need two propositions that essentially state that, with very high probability, certain random simplices have “good properties.”

Before we state them, we remind some basic properties on simplices and convex bodies. We denote by $S(v_0, \dots, v_n)$ the convex hull of the points $v_0, \dots, v_n \in \mathbb{R}^n$ or, in other words, the simplex with vertices v_0, \dots, v_n . It is easy to see that the barycenter/centroid of a simplex $S(v_0, \dots, v_n)$ is given by the geometric mean of the vertices,

$$\text{bar}(S(v_0, \dots, v_n)) = \frac{1}{(n + 1)} \sum_{i=0}^n v_i. \tag{16}$$

Suppose $K \subset \mathbb{R}^n$ is an isotropic convex body and we randomly choose X_1, \dots, X_n in K . The following statement asserts that typically the barycenter of the random simplex $S(0, X_1, \dots, X_n)$ has “small” norm.

Proposition 2.1 *There is an absolute constant $c_1 > 0$ such that for every isotropic convex body $K \subset \mathbb{R}^n$ and $\{X_i\}_{i=1}^n$ independent random vectors uniformly distributed in K then*

$$\mathbb{P} \{ \|\text{bar}(T)\| \leq c_1 L_K \} > 1 - \frac{1}{2} e^{-n}, \tag{17}$$

where T is the random simplex $S(0, X_1, \dots, X_n)$.

Our arguments to prove this proposition are based on the proofs of [2, Theorem 3.1] and [18, Theorem 1.1]. We need to state two lemmas. For elementary background on Orlicz spaces, we refer the reader to [1, Sect. 3.6.2].

The first fact we need, Lemma 2.2, asserts a “good behavior” of the marginals $\langle \cdot, \theta \rangle$, for any direction $\theta \in S^{n-1}$.

Lemma 2.2 *There is an absolute constant $C > 0$ such that for every isotropic convex body $K \subset \mathbb{R}^n$ and every $\theta \in S^{n-1}$, we have*

$$\|\langle \cdot, \theta \rangle\|_{L_{\psi_1}} \leq CL_K. \tag{18}$$

The previous statement is known in the area and is a direct consequence of [1, Lemma 3.5.5] and [1, Theorem 3.5.11].

We also need a classical inequality due to Bernstein about sums of independent random variables (see, for example [1, Theorem 3.5.16]).

Theorem 2.3 (Bernstein inequality) *Let $\{Y_i\}_{i=1}^n$ be a sequence of random variables with mean 0 on some probability space. Assume that Y_i belong to L_{ψ_1} and that $\|Y_i\|_{L_{\psi_1}} \leq M$ for all $i = 1, \dots, n$. Let $\sigma^2 = \frac{1}{n} \sum_{i=1}^n \|Y_i\|_{L_{\psi_1}}^2$. Then, for all $t > 0$,*

$$\mathbb{P} \left\{ \left| \sum_{i=1}^n Y_i \right| > tn \right\} \leq e^{-Dn \min\{\frac{t^2}{\sigma^2}, \frac{t}{M}\}}, \tag{19}$$

for some absolute constant $D > 0$.

We are now ready to give a proof of Proposition 2.1.

Proof of Proposition 2.1 Let $\{X_i\}_{i=1}^n$ be independent random vectors uniformly distributed on K and let θ be fixed direction in S^{n-1} .

By combining Lemma 2.2 and Theorem 2.3 for the random variables $Y_j := \langle X_j, \theta \rangle$ we have, for all $t > CL_K$,

$$\mathbb{P} \left\{ \left| \left\langle \sum_{i=1}^n X_i, \theta \right\rangle \right| > tn \right\} \leq e^{-n \frac{tD}{CL_K}}.$$

Let \mathcal{N} be a $\frac{1}{2}$ -net on the sphere of cardinality less than or equal to 5^n (see e.g., [1, Lemma 5.2.5]). Then

$$\mathbb{P} \left\{ \left| \left\langle \sum_{i=1}^n X_i, \theta \right\rangle \right| > tn \text{ for some } \theta \in \mathcal{N} \right\} \leq e^{-n(\frac{tD}{CL_K} - \log(5))},$$

and hence

$$\mathbb{P} \left\{ \left| \left\langle \sum_{i=1}^n X_i, \theta \right\rangle \right| \leq tn \text{ for every } \theta \in \mathcal{N} \right\} \geq 1 - e^{-n(\frac{tD}{CL_K} - \log(5))}.$$

Every vector $\vartheta \in S^{n-1}$ can be written in the form $\vartheta = \sum_{j=1}^n \delta_j \theta_j$, with $\theta_j \in \mathcal{N}$ and $0 \leq \delta_j \leq 2^{1-j}$ (see for example the proof of [1, Proposition 5.2.8]).

Observe that

$$\bigcap_{\theta \in \mathcal{N}} \left\{ \left| \left\langle \sum_{i=1}^n X_i, \theta \right\rangle \right| \leq tn \right\} \subset \left\{ \left\| \sum_{i=1}^n X_i \right\| \leq 2tn \right\} = \left\{ \max_{\vartheta \in S^{n-1}} \left| \left\langle \sum_{i=1}^n X_i, \vartheta \right\rangle \right| \leq 2tn \right\}.$$

Indeed, let ϑ be an arbitrary unit vector and suppose that $|\langle \sum_{i=1}^n X_i, \theta \rangle| \leq tn$ for every $\theta \in \mathcal{N}$, then

$$\left| \left\langle \sum_{i=1}^n X_i, \vartheta \right\rangle \right| = \left| \left\langle \sum_{i=1}^n v_i, \sum_{j=1}^{\infty} \delta_j \theta_j \right\rangle \right| \leq \sum_{j=1}^{\infty} \delta_j \left| \left\langle \sum_{i=1}^n X_i, \theta_j \right\rangle \right| \leq 2tn.$$

Thus, for every $t > CL_K$ we have

$$\mathbb{P} \left\{ \left\| \sum_{i=1}^n X_i \right\| \leq 2tn \right\} \geq 1 - e^{-n \left(\frac{t}{CL_K} - \log(5) \right)}.$$

The result now follows by setting $t := \frac{c_1(n+1)L_K}{2n}$, for $c_1 > 0$ sufficiently large. □

The second proposition we need asserts that the simplex $S(0, X_1, \dots, X_n)$ typically has “large volume.”

Proposition 2.4 *There is an absolute constant $c_2 > 0$ such that for every isotropic convex body $K \subset \mathbb{R}^n$ and $\{X_i\}_{i=1}^n$ independent random vectors uniformly distributed in K then*

$$\mathbb{P} \left\{ \text{vol}(S(0, X_1, \dots, X_n)) \geq \frac{c_2^n L_K^n}{n^{\frac{n}{2}}} \right\} > 1 - \frac{1}{2} e^{-n}. \tag{20}$$

A proof of it can be found essentially in the work of Pivovarov [31, Proposition 1]. We include the details for completeness.

Lemma 2.5 [31, Lemma 2] *Let $K \subset \mathbb{R}^n$ an isotropic convex body and X be a random vector uniformly distributed on K . Let $E \subset \mathbb{R}^n$ be a k -dimensional subspace and P_E the orthogonal projection onto E . Then the random variable*

$$Y := \frac{|P_E(X)|}{L_K \sqrt{k}}$$

satisfies

$$\mathbb{E} |Y|^{-\frac{1}{2}} \leq C',$$

where $C' > 0$ is an absolute constant.

Proof of Proposition 2.4 Let $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the linear transformation mapping the canonical basis $\{e_i\}_{i=1}^n$ to $\{X_i\}_{i=1}^n$. We have

$$\text{vol}(S(0, X_1, \dots, X_n)) = \frac{|\det(A)|}{n!}. \tag{21}$$

Set $V_k := \text{span}\{X_1, \dots, X_k\}$ and $Y_k = \frac{|P_{V_k^\perp} X_k|}{L_K \sqrt{n-k+1}}$. Note that by Lemma 2.5 if X_1, \dots, X_{k-1} are fixed we have $\mathbb{E}[|Y_k|^{-\frac{1}{2}}] \leq C'$.

Using the fact that

$$|\det(A)| = \|X_1\| \|P_{V_1^\perp}\| \dots \|P_{V_{n-1}^\perp}\| \tag{22}$$

and applying Fubini theorem iteratively, we obtain

$$\mathbb{E} \left[\prod_i^n Y_k^{-\frac{1}{2}} \right] \leq (C')^n. \tag{23}$$

Let $\alpha > 0$ be a constant to be determined. Then by Markov inequality and Eq. (23) we have

$$\begin{aligned} \mathbb{P}(|\det(A)| < \alpha^n L_K^n \sqrt{n!}) &= \mathbb{P} \left(\prod_i^n Y_k < \alpha^n \right) \\ &= \mathbb{P} \left(\prod_i^n Y_k^{-\frac{n}{2}} > \alpha^{-\frac{n}{2}} \right) \\ &\leq \mathbb{E} \left[\prod_i^n Y_k^{-\frac{1}{2}} \right] \alpha^{\frac{n}{2}}. \end{aligned}$$

Setting $\alpha = (eC')^{-2}$ we obtain

$$\mathbb{P} \left(\text{vol}(S(0, X_1, \dots, X_n)) < \frac{\alpha^n L_K^n}{\sqrt{n!}} \right) \leq \frac{1}{2} e^{-n}.$$

The result follows by applying Stirling formula. □

Based on the arguments given in the recent paper of Naszódi [28] and with Propositions 2.1 and 2.4 at hand, we can now give a proof of Theorem 1.5.

Proof of Theorem 1.5 Let $K \subset \mathbb{R}^n$ be an isotropic convex body and X_1, \dots, X_n be independent random vectors uniformly distributed on K . Denote by T the simplex $S(0, X_1, \dots, X_n)$ and by u its barycenter, i.e., $u = \frac{1}{n+1} \sum_{i=1}^n X_i$. By Proposition 2.1 there is an absolute constant $c_1 > 0$ such that

$$\mathbb{P} \{ \|u\| \leq c_1 L_K \} > 1 - \frac{1}{2} e^{-n}. \tag{24}$$

On the other hand, by Proposition 2.4, we know that there is an absolute constant $c_2 > 0$ such that

$$\mathbb{P} \left\{ \text{vol}(T) \geq \frac{c_2^n L_K^n}{n^{\frac{n}{2}}} \right\} > 1 - \frac{1}{2} e^{-n}. \tag{25}$$

By a well-known result of Kannan et al. [20, Theorem 4.1] we have that

$$\sqrt{\frac{n+2}{n}} L_K B_2^n \subset K \tag{26}$$

(note that for the authors the definition of an isotropic convex body is different, that is why the constant L_K in the theorem is missing). Therefore, the vector $w := -\frac{1}{c_1}u$ belongs to K .

It is easy to check that if we apply the homothetic transformation with center w and ratio

$$\lambda = \frac{\|w\|}{\|w - u\|} = \frac{\|w\|}{\|w\| + \|u\|} = \frac{1}{1 + c_1} > 0$$

to the simplex T , we obtain another simplex S with barycenter at the origin (see Fig. 2) such that

$$\text{vol}(S) \geq \lambda^n \text{vol}(T) \geq \lambda^n \cdot \frac{c_2^n L_K^n}{n^{\frac{n}{2}}}. \tag{27}$$

Denote by $\bar{X} := \frac{1}{n+1} \sum_{i=1}^n X_i$. Therefore, the function $f_n : \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_n \rightarrow \mathcal{S}_n^n$ we are looking for can be defined by

$$\begin{aligned} f_n(X_1, \dots, X_n) &:= \varphi(S(0, X_1, \dots, X_n)) \\ &= \frac{1}{1 + c_1} S(-\bar{X}, X_1 - \bar{X}, \dots, X_n - \bar{X}). \end{aligned}$$

This concludes the proof. □

2.1 Deduction of Theorem 1.3 and Its Correct Asymptotic Behavior

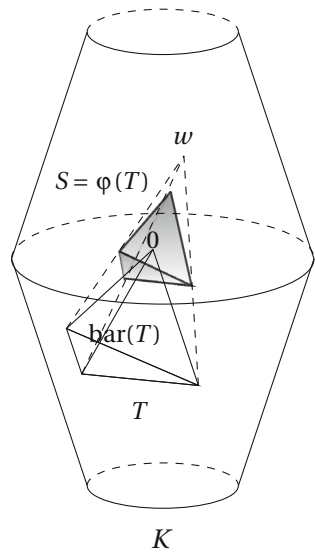
In this section we show how to deduce our main result, Theorem 1.3 from Theorem 1.4. We also show that the volume ratio, up to the absolute constants, is sharp.

We start with a well-known lemma.

Lemma 2.6 *For any simplex $S \subset \mathbb{R}^n$ with barycenter at the origin, we have*

$$M(S) = \frac{(n+1)^{(n+1)}}{(n!)^2}. \tag{28}$$

Fig. 2 Construction involved in the proof of Theorem 1.5



Unfortunately, we could not find an exact reference of the previous lemma. We include a sketch of its proof: the Mahler product of a simplex is invariant under linear transformations, then it is possible to compute it using a particular example (all simplices belong to the same equivalence class). In particular, let $\{e_i\}_{i=1}^n$ be the canonical basis and consider the simplex $S := S(e_1, e_2, \dots, e_n, -\sum_{i=1}^n e_i)$, then its volume is $(n + 1)/n!$. On the other hand, its polar S° is the simplex $S(v_0, v_1, \dots, v_n)$ where $v_0 = \sum_{i=1}^n e_i$ and $v_j = v_0 - (n + 1)e_j$ for $1 \leq j \leq n$, whose volume is $(n + 1)^n/n!$.

We can now give a proof of Theorem 1.3.

Proof Let $K \subset \mathbb{R}^n$ be an arbitrary convex set with barycenter at the origin. By the Rogers–Shephard inequality [1, Theorem 1.5.2], the centrally symmetric so-called difference body $D(K) = K - K$ contains K and fulfills

$$\left(\frac{\text{vol}(D(K))}{\text{vol}(K)}\right)^{1/n} \leq 4. \tag{29}$$

By Eq.(13) applied to the body $D(K)^\circ$ there is a simplex with barycenter at the origin $T \subset D(K)^\circ$ such that

$$\left(\frac{\text{vol}(D(K)^\circ)}{\text{vol}(T)}\right)^{1/n} \leq c \frac{\sqrt{n}}{L_{D(K)^\circ}}, \tag{30}$$

where $c > 0$ is an absolute constant.

Consider S the simplex T° . It is not difficult to see that S has also barycenter at the origin and obviously $S \supset D(K)$. Now,

$$\frac{\text{vol}(S)}{\text{vol}(D(K))} = \frac{\text{vol}(S)\text{vol}(T)}{\text{vol}(D(K))\text{vol}(D(K)^\circ)} \cdot \frac{\text{vol}(D(K)^\circ)}{\text{vol}(T)}. \tag{31}$$

By Lemma 2.6, the Bourgain–Milman inequality [1, Theorem 8.2.2], and Stirling formula we have

$$\left(\frac{\text{vol}(S)\text{vol}(T)}{\text{vol}(D(K))\text{vol}(D(K)^\circ)} \right)^{1/n} \leq c \tag{32}$$

for an absolute constant $c > 0$.

The result now follows immediately from Eqs. (30), (31), (32) and the fact that $D(K) \supset K$ and hence $S \supset K$. \square

As we can see in the following example, the asymptotic behavior of the volume ratio given in Theorem 1.3 cannot be improved.

Example 2.7 (The minimal volume simplex for the Euclidean ball) Let $K := B_2^n$, the Euclidean ball, and $S \supset B_2^n$ the regular simplex circumscribing K . As we can infer from the proof of [1, Theorem 2.4.8(ii)] $K = B_2^n$ is the maximal volume ellipsoid inside S or, in other words, S is in John position.

Let us see that S is the minimal volume simplex containing K . If not, then there is a simplex $T \subset \mathbb{R}^n$ enclosing the ball with $\text{vol}(T) < \text{vol}(S)$. Consider the linear transformation $A \in GL(n)$ such that $A(S) = T$; then, $|\det(A)| < 1$. Therefore $A^{-1}(B_2^n)$ is an ellipsoid with volume greater than $\text{vol}(B_2^n)$ inside S , which is a contradiction.

If we compute the volumes (for the simplex it is easier to do it working in \mathbb{R}^{n+1} on the hyperplane $\sum_i x_i = 1$), we have:

$$\text{Vol}(S) = \frac{(n+1)^{\frac{n+1}{2}} n^{\frac{n}{2}}}{n!}, \tag{33}$$

$$\text{Vol}(K) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)}. \tag{34}$$

Using Stirling formula we therefore get,

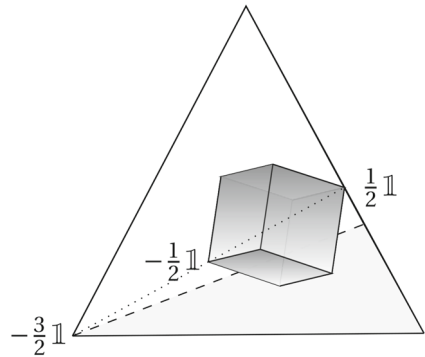
$$\left(\frac{\text{Vol}(S)}{\text{Vol}(K)} \right)^{\frac{1}{n}} \approx \tilde{d}\sqrt{n}, \tag{35}$$

for an absolute constant $\tilde{d} > 0$.

3 The Case of the Cube and a Non-probabilistic Proof of Theorem 1.3

As mentioned, for $n = 2$ the cube has the largest volume ratio (respect to the simplex of minimal volume containing it); for $n = 3$ the same is conjectured. One should expect that a similar phenomenon occurs in high dimensions but, as we can see in the following example, the volume ratio of the cube is uniformly bounded. Moreover, we show that the simplex can be taken with the same barycenter as the cube (Fig. 3).

Fig. 3 The simplex $S\left(-\frac{3}{2}\mathbb{1}, 3e_1 - \frac{1}{2}\mathbb{1}, 3e_2 - \frac{1}{2}\mathbb{1}, 3e_3 - \frac{1}{2}\mathbb{1}\right)$ enclosing the cube $[-\frac{1}{2}, \frac{1}{2}]^3 \subset \mathbb{R}^3$ as in Example 3.1



Example 3.1 Let K be the cube $[-\frac{1}{2}, \frac{1}{2}]^n \subset \mathbb{R}^n$. There is a centered simplex S such that $K \subset S$ and

$$\left(\frac{\text{Vol}(S)}{\text{Vol}(K)}\right)^{\frac{1}{n}} \leq c, \tag{36}$$

for an absolute constant $c > 0$.

Proof Denote by $\mathbb{1}$ the vector in \mathbb{R}^n defined as $\sum_{j=1}^n e_j$. Consider the simplex

$$S := S\left(-\frac{n}{2}\mathbb{1}, ne_1 - \frac{1}{2}\mathbb{1}, ne_2 - \frac{1}{2}\mathbb{1}, \dots, ne_n - \frac{1}{2}\mathbb{1}\right).$$

It is easy to see that $\text{bar}(S) = \text{bar}(K) = 0$. Observe also that the cube $[-\frac{1}{2}, \frac{1}{2}]^n$ is included in the simplex $T := S\left(-\frac{1}{2}\mathbb{1}, ne_1 - \frac{1}{2}\mathbb{1}, ne_2 - \frac{1}{2}\mathbb{1}, \dots, ne_n - \frac{1}{2}\mathbb{1}\right)$. Indeed, all the points that lie in the cube have coordinates greater than or equal to $-\frac{1}{2}$ and their sum is less than or equal to $\frac{n}{2}$. It remains to see that the point $-\frac{1}{2}\mathbb{1}$ belongs to S , but $-\frac{1}{2}\mathbb{1}$ is exactly $t(-\frac{n}{2}\mathbb{1}) + (1-t)\frac{1}{2}\mathbb{1}$ for $t = \frac{2}{n+1}$. An easy computation proves that the volume of S is exactly $\frac{n^n(n+1)}{2n!}$. \square

We end the article giving a non-probabilistic proof of Theorem 1.3. This relies on the Rogers–Shephard inequality, the Dvoretzky–Rogers theorem, and previous estimate for the cube.

Proof of Theorem 1.3 Again, by the Rogers–Shephard inequality [1, Theorem 1.5.2] (see (29)) we can suppose, without loss of generality that K is centrally symmetric. Using a well-known result of Dvoretzky and Rogers [10, Theorem 5A] (see also [33]), there is a centrally symmetric parallelepiped $P \supset K$ such that

$$\left(\frac{\text{vol}(P)}{\text{vol}(K)}\right)^{1/n} \leq c\sqrt{n}, \tag{37}$$

for some absolute constant $c > 0$. The result now follows combining Eq. (37) and the bound given in Example 3.1 for the simplex containing the parallelepiped P (with, of course, (12)). \square

Comparing the result obtained with this technique with (5) and (6), one should note that the isotropic constant is missing (maybe in case the isotropic constant conjecture [1, Conjecture 10.1.7] is false, (5) or (6) could give better estimates for certain bodies).

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References

1. Artstein-Avidan, S., Giannopoulos, A., Milman, V.D.: *Asymptotic Geometric Analysis, Part I*, vol. 202. American Mathematical Society, Providence (2015)
2. Alonso-Gutiérrez, D.: On the isotropy constant of random convex sets. *Proc. Am. Math. Soc.* **136**(9), 3293–3300 (2008)
3. Ball, K.: Volume ratios and a reverse isoperimetric inequality. *J. Lond. Math. Soc.* **2**(2), 351–359 (1991)
4. Barthe, F.: On a reverse form of the Brascamp-Lieb inequality. *Invent. Math.* **134**(2), 335–361 (1998)
5. Brazitikos, S., Giannopoulos, A., Valettas, P., Vritsiou, B.-H.: *Geometry of Isotropic Convex Bodies*, vol. 196. American Mathematical Society, Providence (2014)
6. Blaschke, W.: Über affine geometrie iii: Eine minimumeigenschaft der ellipse. *Berichte über die Verhandlungen der königl. sächs. Gesellschaft der Wissenschaften zu Leipzig* **69**, 3–12 (1917)
7. Bourgain, J.: On the distribution of polynomials on high dimensional convex sets. In: Milman, L.A. (ed.) *Geometric Aspects of Functional Analysis*, pp. 127–137. Springer, New York (1991)
8. Brazitikos, S.: Brascamp-Lieb inequality and quantitative versions of Helly’s theorem. *Mathematika* **63**(1), 272–291 (2017)
9. Chakerian, G.: Minimum area of circumscribed polygons. *Elem. Math.* **28**, 108–111 (1973)
10. Dvoretzky, A., Rogers, C.A.: Absolute and unconditional convergence in normed linear spaces. *Proc. Natl. Acad. Sci.* **36**(3), 192–197 (1950)
11. Gardner, R.J.: *Geometric Tomography*, vol. 1. Cambridge University Press, Cambridge (1995)
12. Giannopoulos, A., Hartzoulaki, M.: On the volume ratio of two convex bodies. *Bull. Lond. Math. Soc.* **34**(06), 703–707 (2002)
13. Giannopoulos, A., Perissinaki, I., Tsolomitis, A.: John’s theorem for an arbitrary pair of convex bodies. *Geom. Dedicata* **84**(1–3), 63–79 (2001)
14. Gross, W.: Über affine geometrie xiii: Eine minimumeigenschaft der ellipse und des ellipsoids. *Ber. Verh. Sächs. Akad. Wiss. Leipz. Math.-Nat. wiss. Kl* **70**, 38–54 (1918)
15. Gruber, P.: *Convex and Discrete Geometry*, vol. 336. Springer, Berlin (2007)
16. Hudelson, M., Klee, V., Larman, D.: Largest j -simplices in d -cubes: some relatives of the Hadamard maximum determinant problem. *Linear Algebra Appl.* **241**, 519–598 (1996)
17. Kanazawa, A.: On the minimal volume of simplices enclosing a convex body. *Arch. Math.* **102**(5), 489–492 (2014)
18. Klartag, B., Kozma, G.: On the hyperplane conjecture for random convex sets. *Israel J. Math.* **170**(1), 253–268 (2009)
19. Klartag, B.: On convex perturbations with a bounded isotropic constant. *Geom. Funct. Anal. GAFA* **16**(6), 1274–1290 (2006)
20. Kannan, R., Lovász, L., Simonovits, M.: Isoperimetric problems for convex bodies and a localization lemma. *Discret. Comput. Geom.* **13**(3–4), 541–559 (1995)
21. Kuperberg, W.: On minimum area quadrilaterals and triangles circumscribed about convex plane regions. *Elem. Math.* **38**, 57–61 (1983)
22. Lassak, M.: On the Banach-Mazur distance between convex bodies. *J. Geom.* **41**, 11–12 (1992)
23. Lassak, M.: Approximation of convex bodies by centrally symmetric bodies. *Geom. Dedicata* **72**(1), 63–68 (1998)
24. Macbeath, A.M.: An extremal property of the hypersphere. In: *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 47, pp. 245–247. Cambridge University Press, Cambridge (1951)

25. Matoušek, Jiří.: Lectures on Discrete Geometry, vol. 108. Springer, New York (2002)
26. McKinney, J.A.M.E.S.R.: On maximal simplices inscribed in a central convex set. *Mathematika* **21**(01), 38–44 (1974)
27. Milman, V.D., Pajor, A.: Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n -dimensional space. In: *Geometric Aspects of Functional Analysis*, pp. 64–104. Springer, New York (1989)
28. Naszódi, M.: Proof of a conjecture of Bárány, Katchalski and Pach. *Discret. Comput. Geom.* **55**(1), 243–248 (2016)
29. Pach, J., Agarwal, P.K.: *Combinatorial Geometry*, vol. 37. Wiley, New York (2011)
30. Pelczynski, A.: Structural theory of Banach spaces and its interplay with analysis and probability. In: *Proceedings of the ICM*, pp. 237–269 (1983)
31. Pivovarov, P.: On determinants and the volume of random polytopes in isotropic convex bodies. *Geom. Dedicata* **149**(1), 45–58 (2010)
32. Paouris, G., Pivovarov, P.: Random ball-polyhedra and inequalities for intrinsic volumes. *Monatshefte für Mathematik* **182**(3), 709–729 (2017)
33. Pelczynski, A., Szarek, S.: On parallelepipeds of minimal volume containing a convex symmetric body in \mathbb{R}^n . In: *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 109, pp. 125–148. Cambridge University Press, Cambridge (1991)
34. Sas, E.: Über eine extremumeigenschaft der ellipsen. *Compos. Math.* **6**, 468–470 (1939)
35. Schneider, R., Weil, W.: *Stochastic and Integral Geometry*. Springer, Berlin (2008)