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Natural symmetric tensor norms

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ABSTRACT

In the spirit of the work of Grothendieck, we introduce and study natural symmetric n-fold tensor norms. These are norms obtained from the projective norm by some *natural* operations. We prove that there are exactly six natural symmetric tensor norms for $n \geqslant 3$, a noteworthy difference with the 2-fold case in which there are four. We also describe the polynomial ideals associated to these natural symmetric tensor norms. Using a symmetric version of a result of Carne, we establish which natural symmetric tensor norms preserve the Banach algebra structure.

1. Introduction

Alexsander Grothendieck's *"Résumé de la théorie métrique des produits tensoriels topologiques"* [15] is considered one of the most influential papers in functional analysis. In this masterpiece, Grothendieck created the basis of what was later known as 'local theory', and exhibited the importance of the use of tensor products in the theory of Banach spaces and operator ideals. As part of his contributions, the *Résumé* contained the list of all *natural* tensor norms. Loosely speaking, these norms come from applying a finite number of *natural* operations to the projective tensor norm. They are obtained by dualization, transposition and by taking left/right projective and injective associates in some order (see Sections 15 and 20 in [7]). Grothendieck proved that there were at most fourteen possible natural norms, but he did not know the exact dominations among them, or if there was a possible reduction on the table of natural norms (this was, in fact, one of the open problems posed in the *Résumé*). This was solved, several years later, thanks to very deep ideas of Gordon and Lewis [14]. All these results are now classical and can be found for example in [7, Section 27] and [8, 4.4.2]. It must be pointed out that one of the strengths of Grothendieck's result is that most of his fourteen tensor norms (at least ten) are *really natural*, since they turn out to be equivalent to the most relevant tensor norms: those related to the ideals of bounded, integral, absolutely r-summing ($r = 1, 2$), r-factorable ($r = 1, 2, \infty$) and 2-dominated operators. These tensor norms appear *naturally* in the theory by their own interest, and it is a remarkable thing that they can be obtained from the projective norm by means of the *natural* operations introduced by Grothendieck.

Motivated by the increasing interest in the theory of symmetric tensor products, we introduce and study natural s-tensor norms of arbitrary order. These are tensor norms defined on symmetric tensor products of Banach spaces which are natural in the following sense: they are obtained from the symmetric projective tensor norm by natural operations (dualization and taking projective and injective associates, see Definition 4.1 below). Among the fourteen non-equivalent natural 2-fold tensor norms, there are exactly four which are symmetric. The s-tensor version of these four tensor norms are, as expected, the only natural ones for symmetric 2-fold tensor products. One of our main results (Theorem 4.2) shows that for $n \geqslant 3$

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we have actually six natural s-tensor norms, a noteworthy difference with the 2-fold case. What we do not have in the n-fold case is the double sense of the word 'natural': at most three among the six obtained tensor norms can be considered *really natural*, if by natural we understand those symmetric tensor norms that naturally appear in the theory. These are the symmetric projective and injective tensor norms and (arguably) the so-called tensor norm η, which appears in relation to extension of polynomials. We then stress that, throughout the article, by natural we just mean those symmetric tensor norms which are obtained from the projective one by the already mentioned operations. In Theorem 4.2 we also describe the maximal ideals of polynomials associated to the six natural norms. For this, we study the adjoint of a composition ideal (Proposition 3.3) and give, in Theorem 3.1, a characterization of the maximal polynomial ideals associated to the injective/projective associates of an s-tensor norm.

The 2-fold tensor norm w_2' is one of the fourteen Grothendieck's natural tensor norms, since it is equivalent to $\backslash\varepsilon_2/$ (see [7, 20.17]). In fact, it is also equivalent to $\backslash/\pi_2\backslash/$. The same equivalence holds for the analogous 2-fold s-tensor norms but, when we pass to n-fold tensor products with $n \geqslant 3$, the s-tensor norms $\backslash/\pi_{n,s}\backslash/$ and $\backslash\varepsilon_{n,s}/$ are no longer equivalent. They are not only non-equivalent as tensor norms: we show in Theorem 4.6 that these norms do not coincide in the symmetric tensor product of any infinite dimensional normed space (the same holds for the norms $/\pi_{n,s}\backslash$ and $/\backslash\varepsilon_{n,s}/\backslash$). In other words, we can say that w_2' splits into two completely different s-tensor norms when passing to tensor products of order $n \geqslant 3$. One may wonder which of these s-tensor norms of high order is the most natural extension of the 2-fold symmetric analogue of w_2'. We will see that, surprisingly, two characteristic properties of w_2' are, in a sense, a consequence of it being equivalent to $\backslash/\pi_{2,s}\backslash/$ rather than to the most simple $\backslash\varepsilon_{2,s}/$. The first property we consider is the relationship with its adjoint s-tensor norm. The second is related to the preservation of the Banach algebra structure.

Carne showed in [6] that there are exactly four natural 2-fold tensor norms that preserve the Banach algebra structure, two of which are symmetric: π_2 and $w_2' \sim \backslash\varepsilon_2/$. Based on his work we establish in Section 5 which natural s-tensor norms preserve the Banach algebra structure. We show that these are $\pi_{n,s}$ and $\backslash/\pi_{n,s}\backslash/$. Thus, one may think that $\backslash/\pi_{n,s}\backslash/$ is the natural extension of the symmetric analogue of w_2' to tensor norms of higher orders.

All our results on s-tensor norms have their analogues for symmetric tensor norms on full tensor products. We chose to present the results for symmetric tensor products because in the full case proofs are similar and sometimes simpler.

We refer to [7] for the theory of 2-fold tensor norms and operator ideals, and to [9,10,12,13] for symmetric and full tensor products and polynomial ideals.

2. Preliminaries

In this section we recall some definitions and results on the theory of symmetric tensor products and Banach polynomial ideals.

Let $\varepsilon_{n,s}$ and $\pi_{n,s}$ stand for the injective and projective symmetric tensor norms of order n respectively. We say that β is an s-tensor norm of order n if β assigns to each Banach space E a norm $\beta(\,.\,;\otimes^{n,s}E)$ on the n-fold symmetric tensor product $\otimes^{n,s}E$ such that

(1) $\varepsilon_{n,s} \leqslant \beta \leqslant \pi_{n,s}$ on $\otimes^{n,s}E$.
(2) $\|\otimes^{n,s}T : \otimes_\beta^{n,s}E \to \otimes_\beta^{n,s}F\| \leqslant \|T\|^n$ for each operator $T \in \mathcal{L}(E,F)$.

The second property is usually referred to as the *metric mapping property*, and it can be seen that the inequality is actually an equality. The s-tensor norm β is said to be finitely generated if for all $E \in BAN$ (the class of all Banach spaces) and $z \in \otimes^{n,s}E$

$$\beta\big(z, \otimes^{n,s}E\big) = \inf\big\{\alpha\big(z, \otimes^{n,s}M\big):\ M \in FIN(E),\ z \in \otimes^{n,s}M\big\},$$

where $FIN(E)$ denotes the class of all finite dimensional subspaces of E. In this article we will only work with finitely generated tensor norms and, therefore, all tensor norms will be assumed to be so.

For β an s-tensor norm of order n, its dual tensor norm β' is defined on FIN by

$$\otimes_{\beta'}^{n,s}M \overset{1}{:=} \big(\otimes_\beta^{n,s}M'\big)'$$

and extended to BAN as

$$\beta'\big(z, \otimes^{n,s}E\big) := \inf\big\{\beta'\big(z, \otimes^{n,s}M\big):\ M \in FIN(E),\ z \in \otimes^{n,s}M\big\}.$$

In other words, it is extended to BAN in the unique way that makes it finitely generated.

Similarly, a tensor norm α of order n assigns to every n-tuple of Banach spaces $(E_1,\dots,E_n)$ a norm $\alpha(\,.\,;\bigotimes_{i=1}^n E_i)$ on the n-fold (full) tensor product $\bigotimes_{i=1}^n E_i$ such that

(1) $\varepsilon_n \leqslant \alpha \leqslant \pi_n$ on $\bigotimes_{i=1}^n E_i$.
(2) $\|\bigotimes_{i=1}^n T_i : (\bigotimes_{i=1}^n E_i, \alpha) \to (\bigotimes_{i=1}^n F_i, \alpha)\| \leqslant \|T_1\|\cdots\|T_n\|$ for each set of operators $T_i \in \mathcal{L}(E_i,F_i)$, $i = 1,\dots,n$.

Here, ε_n and π_n stand for the injective and projective full tensor norms of order n respectively.

We often call these tensor norms "full tensor norms", in the sense that they are defined on the full tensor product, to distinguish them from the s-tensor norms, that are defined on symmetric tensor products. The full tensor norm α is finitely generated if for all $E_i \in BAN$ and z in $\bigotimes_{i=1}^n E_i$

$$\alpha\left(z, \bigotimes_{i=1}^n E_i\right) := \inf\left\{\alpha\left(z, \bigotimes_{i=1}^n M_n\right):\ M_i \in FIN(E_i),\ i = 1, \ldots, n,\ z \in \bigotimes_{i=1}^n M_i\right\}.$$

If α is a full tensor norm of order n, then the dual tensor norm α' is defined on FIN by

$$\left(\bigotimes_{i=1}^n M_i, \alpha'\right) \stackrel{1}{:=} \left[\left(\bigotimes_{i=1}^n M_i', \alpha\right)\right]'$$

and on BAN by

$$\alpha'\left(z, \bigotimes_{i=1}^n E_i\right) := \inf\left\{\alpha'\left(z, \bigotimes_{i=1}^n M_n\right):\ M_i \in FIN(E_i)\ (i = 1, \ldots, n),\ z \in \bigotimes_{i=1}^n M_i\right\}.$$

Let us recall some definitions on the theory of Banach polynomial ideals [12]. A *Banach ideal of continuous scalar valued n-homogeneous polynomials* is a pair $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ such that:

(i) $\mathcal{Q}(E) = \mathcal{Q} \cap \mathcal{P}^n(E)$ is a linear subspace of $\mathcal{P}^n(E)$ and $\|\cdot\|_{\mathcal{Q}(E)}$ (the restriction of $\|\cdot\|_{\mathcal{Q}}$ to $\mathcal{Q}(E)$) is a norm which makes $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}(E)})$ a Banach space.
(ii) If $T \in \mathcal{L}(E_1, E)$, $p \in \mathcal{Q}(E)$ then $p \circ T \in \mathcal{Q}(E_1)$ and

$$\|p \circ T\|_{\mathcal{Q}(E_1)} \leqslant \|P\|_{\mathcal{Q}(E)} \|T\|^n.$$

(iii) $z \mapsto z^n$ belongs to $\mathcal{Q}(\mathbb{K})$ and has norm one.

Let $(\mathcal{Q}, \|\cdot\|_{\mathcal{Q}})$ be the Banach ideal of continuous scalar valued n-homogeneous polynomials and, for $p \in \mathcal{P}^n(E)$, define

$$\|p\|_{\mathcal{Q}^{max}(E)} := \sup\left\{\|p|_M\|_{\mathcal{Q}(M)}:\ M \in FIN(E)\right\} \in [0, \infty].$$

The maximal hull of $\mathcal{Q}$ is the ideal given by $\mathcal{Q}^{max} := \{p \in \mathcal{P}^n:\ \|p\|_{\mathcal{Q}^{max}} < \infty\}$. An ideal $\mathcal{Q}$ is said to be maximal if $\mathcal{Q} \stackrel{1}{=} \mathcal{Q}^{max}$.
Also, for $q \in \mathcal{P}^n$ we define

$$\|q\|_{\mathcal{Q}^*} := \sup\left\{\left|\langle q|_M, p\rangle\right| M \in FIN(E),\ \|p\|_{\mathcal{Q}(M')} \leqslant 1\right\} \in [0, \infty].$$

We will denote $\mathcal{Q}^*$ the class of all polynomials q such that $\|q\|_{\mathcal{Q}^*} < \infty$.
The *s-tensor norm γ* associated to the Banach polynomial ideal $\mathcal{Q}$ is the unique (finitely generated) tensor norm satisfying

$$\mathcal{Q}(M) \stackrel{1}{=} \otimes_{\gamma}^{n,s} M,$$

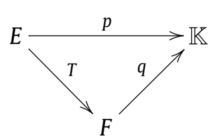
for every finite dimensional space M. The polynomial representation theorem asserts that, if $\mathcal{Q}$ is maximal, then we have

$$\mathcal{Q}(E) \stackrel{1}{=} \left(\widetilde{\otimes}_{\gamma'}^{n,s} E\right)',$$

for every Banach space E [13, 3.2]. It is not difficult to prove that $(\mathcal{Q}^*, \|\ \|_{\mathcal{Q}^*})$ is a maximal Banach ideal of continuous n-homogeneous polynomials. Moreover, if γ is the s-tensor norm associated to the ideal $\mathcal{Q}$ then γ' is the one associated to $\mathcal{Q}^*$ and we have

$$\mathcal{Q}^*(E) \stackrel{1}{=} \left(\widetilde{\otimes}_{\gamma}^{n,s} E\right)'.$$

We will sometimes denote by $\mathcal{Q}_\beta$ the maximal Banach ideal of β-continuous n-homogeneous polynomials, that is, $\mathcal{Q}_\beta(E) := (\widetilde{\otimes}_\beta^{n,s} E)'$. We observe that, with this notation, $\mathcal{Q}_\beta$ is the unique maximal polynomial ideal associated to the s-tensor norm β'.
Let $(\mathfrak{A}, \|\ \|_{\mathfrak{A}})$ be a Banach ideal of operators. The composition ideal $\mathcal{Q} \circ \mathfrak{A}$ is defined in the following way: a polynomial p belongs to $\mathcal{Q} \circ \mathfrak{A}$ if it admits a factorization



(1)

where F is a Banach space, $T \in \mathfrak{A}(E, F)$ and q is in $\mathcal{Q}(F)$. The composition quasi-norm is given by $\|p\|_{\mathcal{Q}\circ\mathfrak{A}} := \inf\{\|q\|_{\mathcal{Q}}\|T\|^n_{\mathfrak{A}}\}$, where the infimum runs over all possible factorizations as in (1). When the quasi-norm $\|\ \|_{\mathcal{Q}\circ\mathfrak{A}}$ is actually a norm, $(\mathcal{Q}\circ\mathfrak{A}, \|\ \|_{\mathcal{Q}\circ\mathfrak{A}})$ forms a Banach ideal of continuous polynomials. All the composition ideals that will be of interest to us in the sequel are normed.

Let us now introduce the notion of quotient ideal. For $p \in \mathcal{P}^n$ we define

$$\|p\|_{\mathcal{Q}\circ\mathfrak{A}^{-1}} := \sup\bigl\{\|p\circ T\|_{\mathcal{Q}}\colon\ T \in \mathfrak{A},\ \|T\|_{\mathfrak{A}} \leqslant 1,\ P\circ T \text{ is defined}\bigr\} \in [0, \infty].$$

We will denote by $\mathcal{Q}\circ\mathfrak{A}^{-1}$ the class of all polynomials p such that $\|p\|_{\mathcal{Q}\circ\mathfrak{A}^{-1}} < \infty$. It is not difficult to prove that $(\mathcal{Q}\circ\mathfrak{A}^{-1}, \|\ \|_{\mathcal{Q}\circ\mathfrak{A}^{-1}})$ is Banach ideal of continuous n-homogeneous polynomials with the property that $p \in \mathcal{Q}\circ\mathfrak{A}^{-1}$ if and only if $p\circ T \in \mathcal{Q}$ for all $T \in \mathfrak{A}$. In other words, $\mathcal{Q}\circ\mathfrak{A}^{-1}$ is the largest ideal satisfying $(\mathcal{Q}\circ\mathfrak{A}^{-1})\circ\mathfrak{A} \subset \mathcal{Q}$.

By $\mathcal{P}^n_f$ we will denote the class of finite type polynomials. We say that a polynomial ideal $\mathcal{Q}$ is *accessible* if the following condition holds: for every Banach space E, $q \in \mathcal{P}^n_f(E)$ and $\varepsilon > 0$, there is a closed finite codimensional space $L \subset E$ and $p \in \mathcal{P}^n(E/L)$ such that $q = p\circ Q^E_L$ (where Q^E_L is the canonical quotient map) and $\|p\|_{\mathcal{Q}} \leqslant (1+\varepsilon)\|q\|_{\mathcal{Q}}$.

Let M be a finite dimensional Banach space. For $p \in \mathcal{P}^n(M)$ and $q \in \mathcal{P}^n(M')$, we denote by $\langle q, p\rangle$ the trace-duality of polynomials, defined for $p = (x')^n$ and $q = x^n$ as

$$\langle p, q\rangle = x'(x)^n,$$

and extended by linearity [9, 1.13].

Finally, a surjective mapping $T : E \to F$ is called *a metric surjection* if

$$\bigl\|Q(x)\bigr\|_F = \inf\bigl\{\|y\|_E\colon\ Q(y) = x\bigr\},$$

for all $x \in E$. As usual, a mapping $I : E \to F$ is called *isometry* if $\|Ix\|_F = \|x\|_E$ for all $x \in E$. We will use the notation $\overset{1}{\twoheadrightarrow}$ and $\overset{1}{\hookrightarrow}$ to indicate a metric surjection or an isometry, respectively. We also write $E \overset{1}{=} F$ if E and F are isometrically isomorphic Banach spaces (i.e. there exists a surjective isometry $I : E \to F$). For a Banach space E with unit ball B_E, we call the mapping $Q_E : \ell_1(B_E) \overset{1}{\twoheadrightarrow} E$ given by

$$Q_E\bigl((a_x)_{x\in B_E}\bigr) = \sum_{x\in B_E} a_x x \tag{2}$$

the *canonical quotient mapping*. Also, we consider the *canonical embedding* $I_E : E \to \ell_\infty(B_{E'})$ given by

$$I_E(x) = \bigl(x'(x)\bigr)_{x'\in B_{E'}}. \tag{3}$$

3. Projective and injective associates of an s-tensor norm

In this section we will define the projective and injective associates of an s-tensor norm and describe their associated maximal Banach ideals of polynomials.

The projective and injective associates (or hulls) of β will be denoted, by extrapolation of the 2-fold full case, as $\backslash\beta/$ and $/\beta\backslash$ respectively. The projective associate of β will be the (unique) smallest projective tensor norm greater than β. Following some ideas from [7, Theorem 20.6] we have

$$\otimes^{n,s}Q_E\colon \otimes^{n,s}_\beta \ell_1(E) \overset{1}{\twoheadrightarrow} \otimes^{n,s}_{\backslash\beta/} E,$$

where $Q_E : \ell_1(B_E) \twoheadrightarrow E$ is the canonical quotient map defined in (2).

The injective associate of β will be the (unique) greatest injective tensor norm smaller than β. As in [7, Theorem 20.7] we get

$$\otimes^{n,s}I_E\colon \otimes^{n,s}_{/\beta\backslash} E \overset{1}{\hookrightarrow} \otimes^{n,s}_\beta \ell_\infty(B_{E'}),$$

where I_E is the canonical embedding (3).

The projective and injective associates for a full tensor norm α can be defined in a similar way and satisfy

$$\left(\bigotimes_{i=1}^n \ell_1(E_i), \alpha\right) \overset{1}{\twoheadrightarrow} \left(\bigotimes_{i=1}^n E_i, \backslash\alpha/\right), \qquad \left(\bigotimes_{i=1}^n E_i, /\alpha\backslash\right) \overset{1}{\hookrightarrow} \left(\bigotimes_{i=1}^n \ell_\infty(B_{E_i'}), \alpha\right).$$

The following duality relations for an s-tensor norm β or a full tensor norm α can be obtained proceeding as in [7, Proposition 20.10]:

$$(/\beta\backslash)' = \backslash\beta'/, \qquad (\backslash\beta/)' = /\beta'\backslash, \qquad (/\alpha\backslash)' = \backslash\alpha'/, \qquad (\backslash\alpha/)' = /\alpha'\backslash. \tag{4}$$

Just as in [7, Corollary 20.8], if E is an $\mathcal{L}_{1,\lambda}$ space for every $\lambda > 1$, then β and $\backslash\beta/$ coincide (isometrically) on $\otimes^{n,s}E$. On the other hand, if E is an $\mathcal{L}_{\infty,\lambda}$ space for every $\lambda > 1$, then β and $/\beta\backslash$ coincide in $\otimes^{n,s}E$. A similar result holds for full tensor norms: if $E_1, \dots, E_n$ are $\mathcal{L}_{1,\lambda}$ spaces for every $\lambda > 1$ then α and $\backslash\alpha/$ are equal on $\bigotimes_{i=1}^n E_i$. On the other hand, if $E_1, \dots, E_n$ are $\mathcal{L}_{\infty,\lambda}$ spaces for every $\lambda > 1$ then α and $/\alpha\backslash$ coincide in $\bigotimes_{i=1}^n E_i$.

It is not difficult to prove that an n-homogeneous polynomial p belongs to $\mathcal{Q}_{\backslash\beta/}(E)$ if and only if $p \circ Q_E \in \mathcal{Q}_\beta(\ell_1(B_E))$. Moreover,

$$\|p\|_{\mathcal{Q}_{\backslash\beta/}(E)} = \|p \circ Q_E\|_{\mathcal{Q}_\beta(\ell_1(B_E))}. \tag{5}$$

On the other hand, an n-homogeneous polynomial p belongs to $\mathcal{Q}_{/\beta\backslash}(E)$ if and only if there exists an n-homogeneous polynomial $\overline{p} \in \mathcal{Q}_\beta(\ell_\infty(B_{E'}))$ such that $\overline{p} \circ I_E = p$ and

$$\|p\|_{\mathcal{Q}_{/\beta\backslash}(E)} = \|\overline{p}\|_{\mathcal{Q}_\beta(\ell_\infty(B_{E'}))}. \tag{6}$$

In other words, $/\beta\backslash$-continuous polynomials are those that can be extended to β-continuous polynomials on $\ell_\infty(B_{E'})$ (or any larger space containing E). As a consequence, the injective associate of the projective s-tensor norm, $/\pi_{n,s}\backslash$, is the predual norm of the ideal of extendible polynomials. Recall that a polynomial $p \in \mathcal{P}^n$ is *extendible* [16] if for any Banach space G containing E there exists $\widetilde{p} \in \mathcal{P}^n(G)$ an extension of p. The Banach polynomial ideal of all extendible polynomials is denoted by $\mathcal{P}_e^n(E)$. For $p \in \mathcal{P}_e^n(E)$, its extendible norm is given by

$$\|P\|_{\mathcal{P}_e^n(E)} = \inf\{C > 0\colon \text{ for all } G \supset E \text{ there is an extension of } P \text{ to } G \text{ with norm} \leqslant C\}.$$

The norm $/\pi_{n,s}\backslash$ usually appears in the literature denoted by η (see [2], and also [16], where this norm is constructed in a different way).

The description of the n-linear forms belonging to $(\bigotimes_{i=1}^n E_i, \backslash\alpha/)'$ or to $(\bigotimes_{i=1}^n E_i, /\alpha\backslash)'$ is analogous to that for polynomials.

The following result describes the maximal Banach ideal of polynomials associated to the projective/injective associates of an s-tensor norm in terms of composition ideals.

Theorem 3.1. *Let β be an s-tensor norm of order n. We have the following identities*:

$$\mathcal{Q}_{/\beta\backslash} \overset{1}{=} \mathcal{Q}_\beta \circ \mathcal{L}_\infty \quad \textit{and} \quad \mathcal{Q}_{\backslash\beta/} \overset{1}{=} \mathcal{Q}_\beta \circ (\mathcal{L}_1)^{-1}.$$

To prove this, we will need a polynomial version of the Cyclic Composition Theorem [7, Theorem 25.4].

Lemma 3.2. *Let $(\mathcal{Q}_1, \| \ \|_{\mathcal{Q}_1})$, $(\mathcal{Q}_2, \| \ \|_{\mathcal{Q}_2})$ be two Banach ideals of continuous n-homogeneous polynomials and $(\mathfrak{A}, \| \ \|_{\mathfrak{A}})$ a Banach operator ideal with $(\mathfrak{A}^{dual}, \| \ \|_{\mathfrak{A}^{dual}})$ right-accessible. If*

$$\mathcal{Q}_1 \circ \mathfrak{A} \subset \mathcal{Q}_2,$$

with $\| \ \|_{\mathcal{Q}_2} \leqslant k \| \ \|_{\mathcal{Q}_1 \circ \mathfrak{A}}$ for some positive constant k, then we have

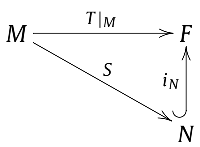
$$\mathcal{Q}_2^* \circ \mathfrak{A}^{dual} \subset \mathcal{Q}_1^*,$$

and $\| \ \|_{\mathcal{Q}_1^*} \leqslant k \| \ \|_{\mathcal{Q}_2^* \circ \mathfrak{A}^{dual}}$.

Proof. Fix $q \in \mathcal{Q}_2^* \circ \mathfrak{A}^{dual}(E)$, $M \in FIN(E)$ and $p \in \mathcal{Q}_1(M')$ with $\|p\|_{\mathcal{Q}_1(M')} \leqslant 1$. For $\varepsilon > 0$, we take $T \in \mathfrak{A}^{dual}(E, F)$ and $q_1 \in \mathcal{Q}_2^*(F)$ such that $q = q_1 \circ T$ and

$$\|q_1\|_{\mathcal{Q}_2^*} \|T\|_{\mathfrak{A}^{dual}}^n \leqslant (1 + \varepsilon)\|q\|_{\mathcal{Q}_2^* \circ \mathfrak{A}^{dual}}.$$

Since $(\mathfrak{A}^{dual}, \| \ \|_{\mathfrak{A}^{dual}})$ is right-accessible, by definition [7, 21.2] there are $N \in FIN(F)$ and $S \in \mathfrak{A}^{dual}(M, N)$ with $\|S\|_{\mathfrak{A}^{dual}} \leqslant (1 + \varepsilon)\|T|_M\|_{\mathfrak{A}^{dual}} \leqslant (1 + \varepsilon)\|T\|_{\mathfrak{A}^{dual}}$ satisfying



(7)

Thus, since the adjoint S^* of S belongs to $\mathfrak{A}(N', M')$, we have

$$\begin{aligned}\big|\langle q|_M, p\rangle\big| &= \big|\langle q_1 \circ T|_M, p\rangle\big| = \big|\langle q_1 \circ i_N \circ S, p\rangle\big| = \big|\langle q_1 \circ i_N, p \circ S^*\rangle\big| \leqslant \|q_1 \circ i_N\|_{\mathcal{Q}_2^*}\big\|p \circ S^*\big\|_{\mathcal{Q}_2}\\ &\leqslant k\|q_1\|_{\mathcal{Q}_2^*}\big\|p \circ S^*\big\|_{\mathcal{Q}_1\circ\mathfrak{A}} \leqslant k\|q_1\|_{\mathcal{Q}_2^*}\|p\|_{\mathcal{Q}_1}\big\|S^*\big\|_{\mathfrak{A}}^n \leqslant k\|q_1\|_{\mathcal{Q}_2^*}\|S\|_{\mathfrak{A}^{dual}}^n \leqslant k(1+\varepsilon)^n\|q_1\|_{\mathcal{Q}_2^*}\|T\|_{\mathfrak{A}^{dual}}^n\\ &\leqslant k(1+\varepsilon)^{n+1}\|q\|_{\mathcal{Q}_2^*\circ\mathfrak{A}^{dual}}.\end{aligned}$$

This holds for every $M \in FIN(E)$ and every $p \in \mathcal{Q}_1(M')$ with $\|p\|_{\mathcal{Q}_1(M')} \leqslant 1$, thus $q \in \mathcal{Q}_1^*$ and $\|q\|_{\mathcal{Q}_1^*} \leqslant k(1+\varepsilon)\|q\|_{\mathcal{Q}_2^*\circ\mathfrak{A}^{dual}}$. Since $\varepsilon > 0$ is arbitrary we get $\|q\|_{\mathcal{Q}_1^*} \leqslant k\|q\|_{\mathcal{Q}_2^*\circ\mathfrak{A}^{dual}}$. □

Notice that the condition of $(\mathfrak{A}^{dual}, \|\ \|_{\mathfrak{A}^{dual}})$ being right-accessible is fulfilled whenever $(\mathfrak{A}, \|\ \|_{\mathfrak{A}})$ is a maximal left-accessible Banach ideal of operators [7, Corollary 21.3].

Proposition 3.3. *Let $(\mathcal{Q}, \|\ \|_{\mathcal{Q}})$ be a Banach ideal of continuous n-homogeneous polynomials and $(\mathfrak{A}, \|\ \|_{\mathfrak{A}})$ a Banach ideal of operators. If $\mathfrak{A}$ is maximal and accessible (or $\mathfrak{A}$ and $\mathfrak{A}^{dual}$ are both right-accessible), and $\mathcal{Q}\circ\mathfrak{A}$ is a Banach ideal of continuous polynomials, then*

$$(\mathcal{Q}\circ\mathfrak{A})^* \overset{1}{=} \mathcal{Q}^* \circ \big(\mathfrak{A}^{dual}\big)^{-1}.$$

Proof. Lemma 3.2 applied to the inclusion $\mathcal{Q}\circ\mathfrak{A} \subset \mathcal{Q}\circ\mathfrak{A}$ implies that $(\mathcal{Q}\circ\mathfrak{A})^* \circ \mathfrak{A}^{dual} \subset \mathcal{Q}^*$. Therefore, $(\mathcal{Q}\circ\mathfrak{A})^* \subset \mathcal{Q}^* \circ (\mathfrak{A}^{dual})^{-1}$ and $\|\ \|_{\mathcal{Q}^*\circ(\mathfrak{A}^{dual})^{-1}} \leqslant \|\ \|_{(\mathcal{Q}\circ\mathfrak{A})^*}$.

For the reverse inclusion we proceed similarly as in proof of Lemma 3.2. Fix $q \in \mathcal{Q}^* \circ (\mathfrak{A}^{dual})^{-1}(E)$, $M \in FIN(E)$ and $p \in \mathcal{Q}\circ\mathfrak{A}(M')$ with $\|p\|_{\mathcal{Q}\circ\mathfrak{A}(M')} \leqslant 1$. For $\varepsilon > 0$, we take $T \in \mathfrak{A}(M', F)$ and $p_1 \in \mathcal{Q}(F)$ such that $p = p_1 \circ T$ and $\|p_1\|_{\mathcal{Q}}\|T\|_{\mathfrak{A}}^n \leqslant (1+\varepsilon)$. Since $(\mathfrak{A}, \|\ \|_{\mathfrak{A}})$ is accessible, there are $N \in FIN(F)$ and $S \in \mathfrak{A}(M', N)$ with

$$\|S\|_{\mathfrak{A}^{dual}} \leqslant (1+\varepsilon)\|T|_M\|_{\mathfrak{A}^{dual}} \leqslant (1+\varepsilon)\|T\|_{\mathfrak{A}}$$

satisfying $T|_M = i_N \circ S$. Note that $S^* \in \mathfrak{A}^{dual}$ and $\|S^*\|_{\mathfrak{A}^{dual}} \leqslant (1+\varepsilon)\|T\|_{\mathfrak{A}}$. Thus, $q|_M \circ (S)^* \in \mathcal{Q}^*$ and $\|q|_M \circ (S)^*\|_{\mathcal{Q}^*} \leqslant (1+\varepsilon)^n\|q\|_{\mathcal{Q}^*\circ(\mathfrak{A}^{dual})^{-1}}\|T\|_{\mathfrak{A}}^n$. Now we have:

$$\begin{aligned}\big|\langle q|_M, p\rangle\big| &= \big|\langle q|_M, p_1 \circ T\rangle\big| = \big|\langle q|_M, p_1 \circ i_N \circ S\rangle\big| \leqslant \big|\langle q|_M \circ S^*, p_1 \circ i_N\rangle\big| \leqslant \big\|q|_M \circ S^*\big\|_{\mathcal{Q}^*}\|p_1 \circ i_N\|_{\mathcal{Q}}\\ &\leqslant (1+\varepsilon)^n\|q\|_{\mathcal{Q}^*\circ(\mathfrak{A}^{dual})^{-1}}\ \|p_1\|_{\mathcal{Q}}\|T\|_{\mathfrak{A}}^n \leqslant (1+\varepsilon)^{n+1}\|q\|_{\mathcal{Q}^*\circ(\mathfrak{A}^{dual})^{-1}}.\end{aligned}$$

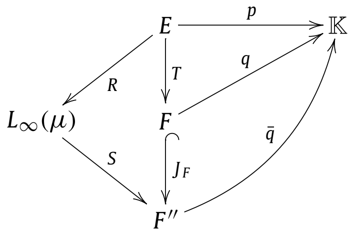
This holds for every $M \in FIN(E)$, every $p \in \mathcal{Q}\circ\mathfrak{A}(M')$ with $\|p\|_{\mathcal{Q}\circ\mathfrak{A}(M')} \leqslant 1$ and every $\varepsilon > 0$. As a consequence, $q \in (\mathcal{Q}\circ\mathfrak{A})^*$ and $\|q\|_{(\mathcal{Q}\circ\mathfrak{A})^*} \leqslant \|q\|_{\mathcal{Q}^*\circ(\mathfrak{A}^{dual})^{-1}}$. □

Now we can prove Theorem 3.1.

Proof of Theorem 3.1. We have already mentioned that any $p \in \mathcal{Q}_{/\beta\backslash}(E)$ extends to a polynomial $\overline{p}$ defined on $\ell_\infty(B_{E'})$ with $\|\overline{p}\|_{\mathcal{Q}_\beta(\ell_\infty(B_{E'}))} = \|p\|_{\mathcal{Q}_{/\beta\backslash}(E)}$. Therefore, p belongs to $\mathcal{Q}_\beta \circ \mathcal{L}_\infty$ and

$$\|p\|_{\mathcal{Q}_\beta\circ\mathcal{L}_\infty} \leqslant \|\overline{p}\|_{\mathcal{Q}_\beta(\ell_\infty(B_{E'}))}\|i\|^n = \|p\|_{\mathcal{Q}_{/\beta\backslash}(E)}.$$

On the other hand, for $p \in \mathcal{Q}_\beta \circ \mathcal{L}_\infty$ and $\varepsilon > 0$ we can take $T \in \mathcal{L}_\infty(E, F)$ and $q \in \mathcal{Q}_\beta(F)$ such that $p = q \circ T$ and $\|q\|_{\mathcal{Q}}\|T\|_{\mathcal{L}_\infty}^n \leqslant (1+\varepsilon)\|p\|_{\mathcal{Q}_\beta\circ\mathcal{L}_\infty}$. We choose $R \in \mathcal{L}(E, L_\infty(\mu))$ and $S \in \mathcal{L}(L_\infty(\mu), F'')$ factoring $J_F \circ T : E \to F''$ with $\|R\|\|S\| \leqslant (1+\varepsilon)\|T\|_{\mathcal{L}_\infty}$. Also, since $\mathcal{Q}_\beta$ is a maximal polynomial ideal, its canonical extension $\bar{q} : F'' \to \mathbb{K}$ belongs to $\mathcal{Q}_\beta$ and satisfies $\|\bar{q}\|_{\mathcal{Q}_\beta} = \|q\|_{\mathcal{Q}_\beta}$ [5]. We then have the following commutative diagram:



Since $\bar{q} \circ S \in \mathcal{Q}_\beta(L_\infty(\mu)) \overset{1}{=} \mathcal{Q}_{/\beta\backslash}(L_\infty(\mu))$ we have

$$\begin{aligned}\|p\|_{\mathcal{Q}_{/\beta\backslash}} &\leqslant \|\bar{q} \circ S\|_{\mathcal{Q}_{/\beta\backslash}}\|R\|^n = \|\bar{q} \circ S\|_{\mathcal{Q}_\beta}\|R\|^n \leqslant \|\bar{q}\|_{\mathcal{Q}_\beta}\|S\|^n\|R\|^n \leqslant (1+\varepsilon)^n\|q\|_{\mathcal{Q}_\beta}\|T\|_{\mathcal{L}_\infty}^n\\ &\leqslant (1+\varepsilon)^{n+1}\|p\|_{\mathcal{Q}_\beta\circ\mathcal{L}_\infty}.\end{aligned}$$

Thus, $\mathcal{Q}_{/\beta\backslash} \overset{1}{=} \mathcal{Q}_\beta \circ \mathcal{L}_\infty$.

Now we show the second identity. First notice that $\mathcal{L}_1 = \mathcal{L}_\infty^{dual}$ (this follows, for example, from Corollary 3 in [7, 17.8] and the information on the table in [7, 27.2]). Since $\mathcal{L}_\infty$ is maximal and accessible [7, Theorem 21.5] and obviously $\mathcal{Q}_{/\beta\backslash}$ is a Banach ideal of continuous polynomials, we can apply Proposition 3.3 to the equality $\mathcal{Q}_{/\beta'\backslash} \overset{1}{=} \mathcal{Q}_{\beta'} \circ \mathcal{L}_\infty$ to obtain $\mathcal{Q}_{\backslash\alpha/} = \mathcal{Q}_\alpha \circ \mathcal{L}_1^{-1}$ with $\| \ \|_{\mathcal{Q}_\alpha \circ \mathcal{L}_1^{-1}} = \| \ \|_{\mathcal{Q}_{\backslash\alpha/}}$. □

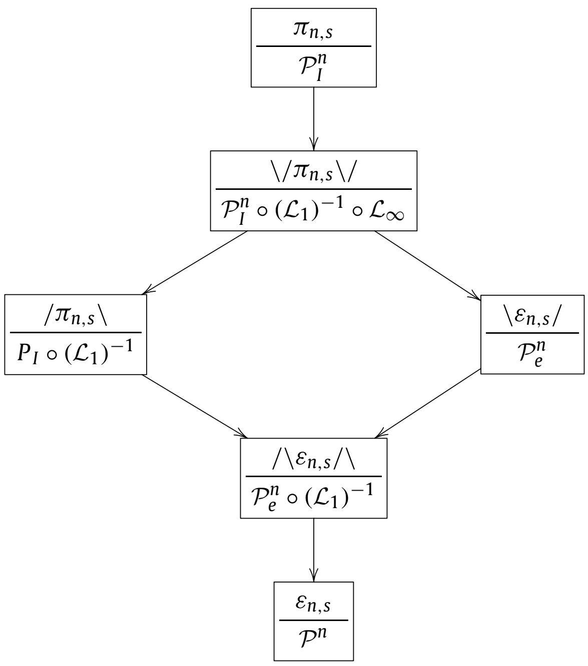
4. Symmetric natural tensor norms of order *n*

In [15] Grothendieck defined natural 2-fold norms as those that can be obtained from π_2 by a finite number of the following operations: dualization, transposition, and taking right injective, left injective, right projective and left projective associate. The aim of this section is to define and study natural symmetric tensor norms of arbitrary order, in the spirit of Grothendieck's norms.

Definition 4.1. Let β be an s-tensor norm of order n. We say that β is a natural s-tensor norm if β is obtained from $\pi_{n,s}$ with a finite number of the operations $\backslash\ /$, $/\ \backslash$ and $'$.

For (full) tensor norms of order 2, there are exactly four natural norms that are symmetric [7, Section 27]. It is easy to show that the same holds for s-tensor norms of order 2 (see the proof of Theorem 4.2). These are $\pi_{2,s}$, $\varepsilon_{2,s}$, $/\pi_{2,s}\backslash$ and $\backslash\varepsilon_{2,s}/$, with the same dominations as in the full case. It is important to mention that, for $n = 2$, $\backslash\varepsilon_{n,s}/$ and $\backslash/\pi_{n,s}\backslash/$ coincide, and so do $/\pi_{n,s}\backslash$ and $/\backslash\varepsilon_{n,s}/\backslash$. However, for $n \geqslant 3$, we have the following.

Theorem 4.2. *For $n \geqslant 3$, there are exactly 6 different natural symmetric s-tensor norms. They can be arranged in the following way:*



(8)

where $\beta \to \gamma$ means that β dominates γ. There are no other dominations than those showed in the scheme. Below each tensor norm we find its associated maximal polynomial ideal.

We recall that $\mathcal{P}_e^n$ stands for the ideal of extendible polynomials (see Section 3). Before we prove the theorem, we need some previous results and definitions. Let α be a full tensor norm of order n. We will denote by $\underline{\alpha}$ the full tensor norm of order $n-1$ given by

$$\underline{\alpha}\left(z, \bigotimes_{i=1}^{n-1} E_i\right) := \alpha(z \otimes 1, E_1 \otimes \cdots \otimes E_{n-1} \otimes \mathbb{C}),$$

where $z \otimes 1 := \sum_{i=1}^m x_1^i \otimes \cdots \otimes x_n^i \otimes 1$, for $z = \sum_{i=1}^m x_1^i \otimes \cdots \otimes x_n^i$ (this definition can be seen as dual to some ideas on [1] and [4]).

Lemma 4.3. *For any tensor norm α, we have: $\underline{(/\alpha\backslash)} = /\underline{\alpha}\backslash$ and $\underline{(\backslash\alpha/)} = \backslash\underline{\alpha}/$. Also, if α and γ are full tensor norms and there exists $C > 0$ such that $\alpha \leqslant C\gamma$, then $\underline{\alpha} \leqslant C\underline{\gamma}$.*

Proof. Let $z \in \bigotimes_{i=1}^n E_i$. For the first statement, if $I_i : E_i \to \ell_\infty(B_{E_i'})$ are the canonical embeddings, we have

$$\begin{aligned}
/\underline{\alpha}\backslash(z, E_1 \otimes \cdots \otimes E_{n-1}) &= \underline{\alpha}\left(\bigotimes_{i=1}^n I_i(z), \ell_\infty(B_{E_1'}) \otimes \cdots \otimes \ell_\infty(B_{E_{n-1}'})\right)\\
&= \alpha\left(\bigotimes_{i=1}^n I_i(z) \otimes 1, \ell_\infty(B_{E_1'}) \otimes \cdots \otimes \ell_\infty(B_{E_{n-1}'}) \otimes \mathbb{C}\right)\\
&= /\alpha\backslash(z \otimes 1, E_1 \otimes \cdots \otimes E_{n-1} \otimes \mathbb{C})\\
&= \underline{(/\alpha\backslash)}(z, E_1 \otimes \cdots \otimes E_{n-1}).
\end{aligned}$$

For the second statement, if $Q_i : \ell_1(B(E_i)) \twoheadrightarrow E_i$ are the canonical quotient mappings, we obtain

$$\begin{aligned}
\backslash\underline{\alpha}/(z, E_1 \otimes \cdots \otimes E_{n-1}) &= \inf_{\{t \,/\, \bigotimes_{i=1}^{n-1} P_i(t)=z\}} \underline{\alpha}\big(t, \ell_1(B_{E_1}) \otimes \cdots \otimes \ell_1(B_{E_{n-1}})\big)\\
&= \inf_{\{t \,/\, \bigotimes_{i=1}^{n-1} P_i(t)=z\}} \alpha\big(t \otimes 1, \ell_1(B_{E_1}) \otimes \cdots \otimes \ell_1(B_{E_n}) \otimes \mathbb{C}\big)\\
&= \inf_{\{t \,/\, (P_1\otimes\cdots\otimes P_{n-1}\otimes id_{\mathbb{C}})(t\otimes 1)=z\otimes 1\}} \alpha\big(t \otimes 1, \ell_1(B_{E_1}) \otimes \cdots \otimes \ell_1(B_{E_{n-1}}) \otimes \mathbb{C}\big)\\
&= \backslash\alpha/(z \otimes 1, E_1 \otimes \cdots \otimes E_{n-1} \otimes \mathbb{C})\\
&= \underline{(\backslash\alpha/)}(z, E_1 \otimes \cdots \otimes E_{n-1}).
\end{aligned}$$

The third statement is immediate. □

Floret in [11] showed that for every s-tensor norm β of order n there exists a full tensor norm $\Phi(\beta)$ of order n which is equivalent to β when restricted on symmetric tensor products (i.e. there is a constant d_n depending only on n such that $d_n^{-1}\Phi(\beta)|_s \leqslant \beta \leqslant d_n\Phi(\beta)|_s$ in $\otimes^{n,s}E$ for every Banach space E). As a consequence, a large part of the isomorphic theory of norms on symmetric tensor products can be deduced from the theory of "full" tensor norms, which is usually easier to handle and has been more studied.

Lemma 4.4. *Let β be an s-tensor norm of order n. Then $\Phi(/\beta\backslash)$ and $/\Phi(\beta)\backslash$ are equivalent s-tensor norms. Also, $\Phi(\backslash\beta/)$ and $\backslash\Phi(\beta)/$ are equivalent s-tensor norms.*

Proof. For simplicity, we consider the case $n = 2$, the proof of the general case being completely analogous. The definition of the injective associate gives

$$E_1 \otimes_{/\Phi(\beta)\backslash} E_2 \overset{1}{\hookrightarrow} \ell_\infty(B_{E_1'}) \otimes_{\Phi(\beta)} \ell_\infty(B_{E_2'}).$$

Take $x_1, \ldots, x_r \in E_1$ and $y_1, \ldots, y_r \in E_2$ and let $I_i : E_i \to \ell_\infty(B_{E_i'})$ be the canonical embeddings (3). Following the notation in [11], we have:

$$\begin{aligned}
/\Phi(\beta)\backslash\left(\sum_{j=1}^r x_j \otimes y_j\right) &= \Phi(\beta)\left(\sum_{j=1}^r I_1(x_j) \otimes I_2(y_j), \ell_\infty(B_{E_1'}) \otimes \ell_\infty(B_{E_2'})\right)\\
&= \sqrt{2}K_2(\beta)^{-1}\beta\left(\sum_{j=1}^r \big(I_1(x_j), 0\big) \vee \big(0, I_2(y_j)\big), \otimes^{2,s}\big\{\ell_\infty(B_{E_1'}) \oplus_2 \ell_\infty(B_{E_2'})\big\}\right)\\
&\asymp \sqrt{2}K_2(\beta)^{-1}\beta\left(\sum_{j=1}^r \big(I_1(x_j), 0\big) \vee \big(0, I_2(y_j)\big), \otimes^{2,s}\big\{\ell_\infty(B_{E_1'}) \oplus_\infty \ell_\infty(B_{E_2'})\big\}\right)\\
&= \sqrt{2}K_2(\beta)^{-1}/\beta\backslash\left(\sum_{j=1}^r \big(I_1(x_j), 0\big) \vee \big(0, I_2(y_j)\big), \otimes^{2,s}\big\{\ell_\infty(B_{E_1'}) \oplus_\infty \ell_\infty(B_{E_2'})\big\}\right)\\
&\asymp \sqrt{2}K_2(\beta)^{-1}/\beta\backslash\left(\sum_{j=1}^r \big(I_1(x_j), 0\big) \vee \big(0, I_2(y_j)\big), \otimes^{2,s}\big\{\ell_\infty(B_{E_1'}) \oplus_2 \ell_\infty(B_{E_2'})\big\}\right)
\end{aligned}$$

$$= \sqrt{2}K_2(\beta)^{-1}/\beta\backslash\left(\sum_{j=1}^{r}(x_j, 0) \vee (0, y_j), \otimes^{2,s}\{E_1 \oplus_2 E_2\}\right)$$

$$= \Phi(/\beta\backslash)\left(\sum_{j=1}^{r} x_j \otimes y_j\right),$$

where $\asymp$ means that the two expressions are equivalent up to universal constants. The second equivalence follows from the first one by duality, since by [11, Theorem 2.3(8)] and (4) we have $\Phi(\backslash\beta/) = \Phi((/\beta'\backslash)') \sim \Phi(/\beta'\backslash)' \sim /\Phi(\beta')\backslash' = \backslash\Phi(\beta')'/ \sim \backslash\Phi(\beta)/$. □

As a consequence of these results we can see that no injective norm β can be equivalent to a projective norm δ. Indeed, if they were equivalent, we would have $\backslash\varepsilon_{n,s}/ \leqslant \backslash\beta/ \leqslant C_1\delta \leqslant C_2\beta \leqslant C_2/\pi_{n,s}\backslash$. Since Φ respects inequalities [11, Theorem 2.3(4)], an application of Lemmas 4.4 and 4.3, together with the obvious identities $\underline{\varepsilon_{n+1}} = \varepsilon_n$, $\underline{\pi_{n+1}} = \pi_n$ would give $\backslash\varepsilon_2/ \sim w_2' \leqslant D/\pi_2\backslash \sim w_2$, a contradiction.

Another consequence is that $\pi_{2,s}$, $\varepsilon_{2,s}$, $/\pi_{2,s}\backslash$ and $\backslash\varepsilon_{2,s}/$ are the non-equivalent natural s-tensor norms for $n = 2$. This follows from the 2-fold result (see [7, Chapter 27]), which states that π_2, ε_2, $/\pi_2\backslash$ and $\backslash\varepsilon_2/$ are the only natural 2-fold tensor norms that are symmetric. So Lemma 4.4 and the properties of Φ give our claim, as well as the following dominations: $\varepsilon_{2,s} \leqslant \backslash\varepsilon_{2,s}/ \leqslant /\pi_{2,s}\backslash \leqslant \pi_{2,s}$.

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. To prove that all the possible natural n-fold s-tensor norms ($n \geqslant 3$) are listed in (8), it is enough to show that $/\backslash/\pi_{n,s}\backslash/\backslash$ coincides with $/\pi_{n,s}\backslash$. From the inequality $\backslash/\pi_{n,s}\backslash/ \leqslant \pi_{n,s}$ we readily obtain $/\backslash/\pi_{n,s}\backslash/\backslash \leqslant /\pi_{n,s}\backslash$. Also, the inequality $\varepsilon_{n,s} \leqslant /\backslash\varepsilon_{n,s}/\backslash$ gives $\backslash\varepsilon_{n,s}/ \leqslant \backslash/\backslash\varepsilon_{n,s}/\backslash/$ and, by duality, we have $/\pi_{n,s}\backslash \leqslant /\backslash/\pi_{n,s}\backslash/\backslash$.

Now we see that the listed norms are all different. First, $/\pi_{n,s}\backslash$ and $\backslash/\pi_{n,s}\backslash/$ cannot be equivalent, since the first one is injective and the second one is projective. Analogously, $\backslash\varepsilon_{n,s}/$ is not equivalent to $/\backslash\varepsilon_{n,s}/\backslash$. Until now, everything works just as in the case $n = 2$. The difference appears when we consider the relationship between $\backslash/\pi_{n,s}\backslash/$ and $\backslash\varepsilon_{n,s}/$: we will see in Theorem 4.6 that $\backslash/\pi_{n,s}\backslash/$ and $\backslash\varepsilon_{n,s}/$ cannot be equivalent on any infinite dimensional Banach space, which is much more than we need. By duality, conclude that the six listed norms in Theorem 4.2 are different.

It is clear that all the dominations presented in (8) hold, so we must show that $/\pi_{n,s}\backslash$ does not dominate $\backslash\varepsilon_{n,s}/$ nor $\backslash\varepsilon_{n,s}/$ dominates $/\pi_{n,s}\backslash$. Note that the inequality $/\pi_{n,s}\backslash \leqslant C\backslash\varepsilon_{n,s}/$ would imply the equivalence between $/\pi_{n,s}\backslash$ and $\varepsilon_{n,s}$ on $\otimes^{n,s}\ell_1$, which is impossible (see [3,17,18]). Finally, if $/\pi_{n,s}\backslash$ dominates $\backslash\varepsilon_{n,s}/$, then we can reason as in the comments after Lemma 4.4 and conclude that $/\pi_2\backslash$ dominates $\backslash\varepsilon_2/$, which contradicts [7, Chapter 27].

The maximal polynomial ideals associated to the natural norms are easily obtained using Theorem 3.1 and the fact that $\mathcal{Q}_{/\beta\backslash}$ and $\mathcal{Q}_{\backslash\gamma/}$ are associated to the norms $\backslash\beta'/$ and $/\gamma'\backslash$ respectively. □

The 2-fold tensor norms π_2 and $\backslash\varepsilon_2/$ (which is equivalent to w_2') share two interesting properties. The first property is that they dominate their dual tensor norm. Indeed, the inequality $\pi_2' = \varepsilon_2 \leqslant \pi_2$ is clear, and we see in [7, 27.2] that w_2 is dominated by w_2' (or, analogously, $/\pi_2\backslash$ is dominated by $\backslash\varepsilon_2/$). The second property is that both π_2 and w_2' preserve the Banach algebra structure [6]. These two properties are enjoyed, of course, by their corresponding 2-fold s-tensor norms (see the proof of Theorem 4.2 for the first one, and Section 5 for the second one). As we have already seen, the n-dimensional analogue of the s-tensor norm $\backslash\varepsilon_{2,s}/$ splits into two non-equivalent ones when passing from tensor products of order 2 to tensor products of order $n \geqslant 3$. Namely, $\backslash\varepsilon_{n,s}/$ and $\backslash/\pi_{n,s}\backslash/$. It is remarkable that the two mentioned properties are enjoyed only by $\backslash/\pi_{n,s}\backslash/$ and not by $\backslash\varepsilon_{n,s}/$, as seen in Theorems 4.2 and 5.3.

Theorem 4.6 below shows that there is no infinite dimensional Banach space E such that $\backslash\varepsilon_{n,s}/$ and $\backslash/\pi_{n,s}\backslash/$ are equivalent in $\otimes^{n,s}E$ for $n \geqslant 3$. This means that the splitting of $\backslash\varepsilon_{n,s}/$ when passing from $n = 2$ to $n \geqslant 3$ is rather drastic. To prove the theorem we need the following proposition.

Proposition 4.5. *Let $\mathcal{Q}$ be a polynomial ideal and β its associated tensor norm. If β is injective then $\mathcal{Q}$ is accessible.*

Proof. Let q be a finite type polynomial on E and choose $(x_j')_{j=1}^r$ in E' such that $q = \sum_{j=1}^r (x_j')^n$. We set $L = \bigcap_{j=1}^r Ker(x_j')$, which is a finite codimensional subspace of E. For each $j = 1, \ldots, r$, let $\overline{x'}_j \in (E/L)'$ be defined by $\overline{x'}_j(\bar{x}) := x_j'(x)$ (where $\bar{x}$ denotes the class of x in E/L). If $Q_E^L : E \to E/L$ is the quotient map and p is the polynomial on E/L given by $p = \sum_{j=1}^r (\overline{x'}_j)^n$, we have $q = p \circ Q_L^E$. Also, since β is injective we have the isometry

$$\otimes^{n,s}\big(Q_L^E\big)' : \otimes_\beta^{n,s}(E/L)' \overset{1}{\hookrightarrow} \otimes_\beta^{n,s}E'.$$

This altogether gives

$$\|p\|_{\mathcal{Q}} = \beta\left(\sum_{j=1}^{r} \otimes^n \bar{x}'_j, \otimes^{n,s}(E/L)'\right) = \beta\left(\otimes^{n,s}\left(Q_L^E\right)'\left(\sum_{j=1}^{r} \otimes^n \overline{x'}_j\right), \otimes^{n,s}E'\right) = \beta\left(\sum_{j=1}^{r} \otimes^n x'_j, \otimes^{n,s}E'\right) = \|q\|_{\mathcal{Q}},$$

which shows the accessibility of $\mathcal{Q}$. □

Theorem 4.6. *For $n \geqslant 3$, $\backslash\varepsilon_{n,s}/$ and $\backslash/\pi_{n,s}\backslash/$ are equivalent in $\otimes^{n,s}E$ if and only if E is finite dimensional. The same happens if $/\pi_{n,s}\backslash$ and $/\backslash\varepsilon_{n,s}/\backslash$ are equivalent on E.*

Proof. We will first prove that if E is infinite dimensional, then $/\pi_{n,s}\backslash$ and $/\backslash\varepsilon_{n,s}/\backslash$ are not equivalent in $\otimes^{n,s}E$. Suppose they are. Then, we have

$$\mathcal{P}_e^n(E) = \left(\otimes^{n,s}_{/\pi_{n,s}\backslash}E\right)' = \left(\otimes^{n,s}_{/\backslash\varepsilon_{n,s}/\backslash}E\right)' = \mathcal{Q}_{/\backslash\varepsilon_{n,s}/\backslash}(E).$$

By the open mapping theorem, there must be a constant $M > 0$ such that $\|p\|_{\mathcal{Q}_{/\backslash\varepsilon_{n,s}/\backslash}(E)} \leqslant M\|p\|_{\mathcal{P}_e^n(E)}$, for every extendible polynomial p on E. If F is a subspace of E, any extendible polynomial on F extends to an extendible polynomial on E with the same extendible norm. Therefore, for every subspace F of E and every extendible polynomial q on F, we have $\|q\|_{\mathcal{Q}_{/\backslash\varepsilon_{n,s}/\backslash}(F)} \leqslant M\|q\|_{\mathcal{P}_e^n(F)}$.

Since E is an infinite dimensional space, by Dvoretzky's theorem it contains $(\ell_2^k)_k$ uniformly. Then there exists a constant $C > 0$ such that for every k and every polynomial q on ℓ_2^k, we have

$$\|q\|_{\mathcal{Q}_{/\backslash\varepsilon_{n,s}/\backslash}(\ell_2^k)} \leqslant C\|q\|_{\mathcal{P}_e^n(\ell_2^k)}.$$

Since the ideal of extendible polynomials is maximal (it is dual to an s-tensor norms), we deduce that

$$\mathcal{P}_e^n(\ell_2) \subset \mathcal{Q}_{/\backslash\varepsilon_{n,s}/\backslash}(\ell_2). \tag{9}$$

Let us show that this is not true. Since $/\backslash\varepsilon_{n,s}/\backslash$ is injective and we have an inclusion $\ell_2 \hookrightarrow L_1[0,1]$, each $p \in \mathcal{Q}_{/\backslash\varepsilon_{n,s}/\backslash}(\ell_2)$ can be extended to a $/\backslash\varepsilon_{n,s}/\backslash$-continuous polynomial $\widetilde{p}$ on $L_1[0,1]$. Now, $\varepsilon_{n,s}$ coincides with $\backslash\varepsilon_{n,s}/$ on $L_1[0,1]$, which is in turn dominated by $/\backslash\varepsilon_{n,s}/\backslash$. Therefore, the polynomial $\widetilde{p}$ is actually $\varepsilon_{n,s}$-continuous or, in other words, integral. Since $\widetilde{p}$ extends p, the latter polynomial must also be integral, and we have shown that $\mathcal{Q}_{/\backslash\varepsilon_{n,s}/\backslash}(\ell_2)$ is contained in $\mathcal{P}_I^n(\ell_2)$. But it is shown in [3,17,18] that there are always extendible non-integral polynomials on any infinite dimensional Banach space, so (9) cannot hold. This contradiction shows that $/\pi_{n,s}\backslash$ and $/\backslash\varepsilon_{n,s}/\backslash$ cannot be equivalent on E.

Now we will show that $\backslash\varepsilon_{n,s}/$ and $\backslash/\pi_{n,s}\backslash/$ are not equivalent in $\otimes^{n,s}E$, for any infinite dimensional Banach space E. Suppose they are. By duality, we have $\mathcal{Q}_{\backslash\varepsilon_{n,s}/} = \mathcal{Q}_{\backslash/\pi_{n,s}\backslash/}$ with equivalent norms. Proposition 4.5 ensures that the polynomial ideals $\mathcal{Q}_{\backslash\varepsilon_{n,s}/}$, $\mathcal{Q}_{\backslash/\pi_{n,s}\backslash/}$ are both accessible, since they are associated to the injective norms $/\pi_{n,s}\backslash$ and $/\backslash\varepsilon_{n,s}/\backslash$ respectively. Thus, by [10, Proposition 3.6] we have:

$$\widetilde{\otimes}^{n,s}_{/\pi_{n,s}\backslash}E' \overset{1}{\hookrightarrow} \mathcal{Q}_{\backslash\varepsilon_{n,s}/}(E) \quad \text{and} \quad \widetilde{\otimes}^{n,s}_{/\backslash\varepsilon_{n,s}/\backslash}E' \overset{1}{\hookrightarrow} \mathcal{Q}_{\backslash/\pi_{n,s}\backslash/}(E).$$

But this implies that $/\pi_{n,s}\backslash$ and $/\backslash\varepsilon_{n,s}/\backslash$ are equivalent in $\otimes^{n,s}E'$, which is impossible by the already proved first statement of the theorem. □

5. s-Tensor norms preserving the Banach algebra structure

Carne in [6] described the natural 2-fold tensor norms that preserve the Banach algebra structure. In this section we will show that $\pi_{n,s}$ and $\backslash/\pi_{n,s}\backslash/$ are the only natural s-tensor norms that preserve the algebra structure.

For a given Banach algebra A we will denote $m(A): A \otimes_{\pi_2} A \to A$ the map induced by the multiplication $A \times A \to A$. The following theorem is a symmetric version of Carne [6, Theorem 1]. Its proof is obtained by adapting the one in [6] for the symmetric setting.

Theorem 5.1. *For an s-tensor norm β of order n the following conditions are equivalent.*

(1) *If A is Banach algebra, the n-fold symmetric tensor product $\widetilde{\otimes}^{n,s}_{\beta}A$ is a Banach algebra with the natural algebra structure.*
(2) *For all Banach spaces E and F there is a natural continuous linear map*

$$f : \left(\otimes^{n,s}_{\beta}E\right) \otimes_{\pi_2} \left(\otimes^{n,s}_{\beta}F\right) \to \left(\otimes^{n,s}_{\beta}(E \otimes_{\pi_2} F)\right)$$

with

$$f\left(\left(\otimes^n x\right) \otimes \left(\otimes^n y\right)\right) = \otimes^n(x \otimes y).$$

(3) *For all Banach spaces E and F there is a natural continuous map*

$$g : \big(\otimes_{\beta'}^{n,s}(E \otimes_{\varepsilon_2} F)\big) \to \big(\otimes_{\beta'}^{n,s} E\big) \otimes_{\varepsilon_2} \big(\otimes_{\beta'}^{n,s} F\big)$$

given by

$$g\big(\otimes^n (x \otimes y)\big) = \big(\otimes^n x\big) \otimes \big(\otimes^n y\big).$$

(4) *For all Banach spaces E and F there is a natural continuous map*

$$h : \otimes_{\beta'}^{n,s} \mathcal{L}(E, F) \to \mathcal{L}\big(\otimes_{\beta}^{n,s} E, \otimes_{\beta'}^{n,s} F\big),$$

with

$$h\big(\otimes^n T\big)\big(\otimes^n x\big) = \otimes^n (Tx).$$

If one, hence all, of the above holds, then there are constants c_1, c_2, c_3, c_4 so that

(1) $\|m(\widetilde{\otimes}_{\beta}^{n,s} A)\| \leqslant c_1 \|m(A)\|^n$,
(2) $\|f\| \leqslant c_2$ *for all E and F*,
(3) $\|g\| \leqslant c_3$ *for all E and F*,
(4) $\|h\| \leqslant c_4$ *for all E and F*,

and the least values of these four agree.

If the s-tensor norm β preserves the Banach algebra structure, then we will call the common least value of the constants in the theorem, *the Banach algebra constant* of β.

An important comment is in order: if we take $E = F$ and $T = id_E$ in (4), then we obtain $\|h(\otimes^{n,s} id_E)\| \leqslant c_4$. But it is plain that $h(\otimes^n id_E)$ is just $id_{\otimes^{n,s} E}$. Therefore, we have

$$\big\| id_{\otimes^{n,s} E} : \otimes_{\beta}^{n,s} E \to \otimes_{\beta'}^{n,s} E \big\| \leqslant c_4,$$

which means that $\beta' \leqslant c_4 \beta$. So we can state the following remark.

Remark 5.2. If β is an s-tensor norm which preserves the Banach algebra structure, then there is a constant k such that $\beta' \leqslant k\beta$.

The following theorem is the main result of this section. The proof that π_s preserves the Banach algebra structure is similar to the one for π_2 in [6], and we include it for completeness.

Theorem 5.3. *The only natural s-tensor norms of order n which preserve the Banach algebra structure are: $\pi_{n,s}$ and $\backslash/\pi_{n,s}\backslash/$. Furthermore, the Banach algebra constants of both norms are exactly one.*

Proof. It follows from Theorem 4.2 and the previous remark that $\pi_{n,s}$ and $\backslash/\pi_{n,s}\backslash/$ are the only candidates among natural s-tensor norms to preserve the Banach algebra structure.

First we prove that π_s preserves the Banach algebra structure. By Theorem 5.1, it is enough to show, for any pair of Banach spaces E and F, that the mapping

$$f : \big(\otimes_{\pi_{n,s}}^{n,s} E\big) \otimes_{\pi_2} \big(\otimes_{\pi_{n,s}}^{n,s} F\big) \to \big(\otimes_{\pi_{n,s}}^{n,s} (E \otimes_{\pi_2} F)\big),$$

defined by

$$f\big(\big(\otimes^n x\big) \otimes \big(\otimes^n y\big)\big) = \otimes^n (x \otimes y),$$

has norm less than or equal to one. Fix $\varepsilon > 0$. Given $w \in (\otimes^{n,s} E) \otimes (\otimes^{n,s} F)$, we can write it as

$$w = \sum_{i=1}^{r} u_i \otimes v_i,$$

with

$$\sum_{i=1}^{r} \pi_{n,s}(u_i)\pi_{n,s}(v_i) \leqslant \pi_2(w)(1 + \varepsilon)^{1/3}.$$

Also, for each $i = 1, \dots, r$ we write u_i and v_i as

$$u_i = \sum_{j=1}^{J(i)} \otimes^n x_j^i \in \otimes^{n,s} E, \qquad v_i = \sum_{k=1}^{K(i)} \otimes^n y_k^i \in \otimes^{n,s} F,$$

with

$$\sum_{j=1}^{J(i)} \|x_j^i\|^n \leqslant \pi_{n,s}(u_i)(1+\varepsilon)^{1/3}, \qquad \sum_{k=1}^{K(i)} \|y_k^i\|^n \leqslant \pi_{n,s}(v_i)(1+\varepsilon)^{1/3}.$$

We have

$$f(w) = \sum_{i=1}^{r} \sum_{\substack{1 \leqslant j \leqslant J(i) \\ 1 \leqslant k \leqslant K(i)}} \otimes^n \big(x_j^i \otimes y_k^i\big),$$

and then

$$\begin{aligned} \pi_{n,s}\big(f(w)\big) &\leqslant \sum_{i=1}^{r} \sum_{\substack{1 \leqslant j \leqslant J(i) \\ 1 \leqslant k \leqslant K(i)}} \pi_2\big(x_j^i \otimes y_k^i\big)^n = \sum_{i=1}^{r} \sum_{\substack{1 \leqslant j \leqslant J(i) \\ 1 \leqslant k \leqslant K(i)}} \|x_j^i\|^n \|y_k^i\|^n = \sum_{i=1}^{r} \Bigg(\sum_{j \leqslant J(i)} \|x_j^i\|^n \Bigg) \Bigg(\sum_{k \leqslant K(i)} \|y_k^i\|^n \Bigg) \\ &= \sum_{i=1}^{r} \pi_{n,s}(u_i)(1+\varepsilon)^{1/3} \pi_{n,s}(v_i)(1+\varepsilon)^{1/3} = (1+\varepsilon)^{2/3} \sum_{i=1}^{r} \pi_2(u_i)\pi_2(v_i) \leqslant (1+\varepsilon)\pi(w). \end{aligned}$$

From this we conclude that $\|f\| \leqslant 1$.

To prove that $\backslash/\pi_{n,s}\backslash/$ preserves the Banach algebra structure we need two technical lemmas.

Lemma 5.4. *Let Y and Z be Banach spaces. The operator*

$$\phi : \otimes^{n,s}_{/\pi_{n,s}\backslash} \mathcal{L}\big(\ell_1(B_Y), Z\big) \to \mathcal{L}\big(\otimes^{n,s}_{/\pi_{n,s}\backslash} \ell_1(B_Y), \otimes^{n,s}_{/\pi_{n,s}\backslash} Z\big),$$

given by

$$\phi\big(\otimes^n T\big)\big(\otimes^n u\big) = \otimes^n T u,$$

has norm less than or equal to one.

Proof. The mapping

$$\begin{aligned} &\mathcal{L}\big(\ell_1(B_Y), \ell_\infty(B_{Z'})\big) \to \mathcal{L}\big(\otimes^{n,s}_{/\pi_{n,s}\backslash} \ell_1(B_Y), \otimes^{n,s}_{/\pi_{n,s}\backslash} Z\big) \\ &T \mapsto \otimes^n T \end{aligned}$$

is an n-homogeneous polynomial, which has norm one by the metric mapping property of the norm $/\pi_{n,s}\backslash$. As a consequence, its linearization is a norm one operator from $\otimes^{n,s}_{/\pi_{n,s}\backslash} \mathcal{L}(\ell_1(B_Y), \ell_\infty(B_{Z'}))$ to $\mathcal{L}(\otimes^{n,s}_{/\pi_{n,s}\backslash} \ell_1(B_Y), \otimes^{n,s}_{/\pi_{n,s}\backslash} Z)$. Since $\mathcal{L}(\ell_1(B_Y), \ell_\infty(B_{Z'}))$ is an $\mathcal{L}_\infty$ space we have

$$\otimes^{n,s}_{/\pi_{n,s}\backslash} \mathcal{L}\big(\ell_1(B_Y), \ell_\infty(B_{Z'})\big) \overset{1}{=} \otimes^{n,s}_{\pi_{n,s}} \mathcal{L}\big(\ell_1(B_Y), \ell_\infty(B_{Z'})\big).$$

This shows that the canonical mapping

$$\otimes^{n,s}_{/\pi_{n,s}\backslash} \mathcal{L}\big(\ell_1(B_Y), \ell_\infty(B_{Z'})\big) \to \mathcal{L}\big(\otimes^{n,s}_{/\pi_{n,s}\backslash} \ell_1(B_Y), \otimes^{n,s}_{/\pi_{n,s}\backslash} \ell_\infty(B_{Z'})\big)$$

has norm one.

On the other hand, the following diagram commutes

$$\begin{array}{ccc} \otimes^{n,s}_{/\pi_{n,s}\backslash} \mathcal{L}(\ell_1(B_Y), \ell_\infty(B_{Z'})) & \longrightarrow & \mathcal{L}(\otimes^{n,s}_{/\pi_{n,s}\backslash} \ell_1(B_Y), \otimes^{n,s}_{/\pi_{n,s}\backslash} \ell_\infty(B_{Z'})) \\ \uparrow & & \uparrow \\ \otimes^{n,s}_{/\pi_{n,s}\backslash} \mathcal{L}(\ell_1(B_Y), Z) & \xrightarrow{\ \phi\ } & \mathcal{L}(\otimes^{n,s}_{/\pi_{n,s}\backslash} \ell_1(B_Y), \otimes^{n,s}_{/\pi_{n,s}\backslash} Z) \end{array}$$

Here the vertical arrows are the natural inclusion, which are actually isometries since the norm $/\pi_{n,s}\backslash$ is injective. The horizontal arrow above is the canonical mapping whose norm was shown to be one. Therefore, the norm of ϕ must be less than or equal to one. $\square$

Before we state our next lemma, we observe that linear operators from X_1 to $\mathcal{L}(X_2, X_3)$ identify (isometrically) with bilinear operators from $X_1 \times X_2$ to X_3 and, consequently, with linear operators from $X_1 \otimes_\pi X_2$ to X_3. The isometry is given by

$$\begin{aligned} &\mathcal{L}\big(X_1, \mathcal{L}(X_2, X_3)\big) \to \mathcal{L}(X_1 \otimes_\pi X_2, X_3) \\ &T \mapsto B_T, \end{aligned} \tag{10}$$

where $B_T(x_1 \otimes x_2) = T(x_1)(x_2)$.

Lemma 5.5. *Let E and F be Banach spaces. The operator*

$$\rho : \big(\otimes^{n,s}_{/\pi_{n,s}\backslash} \ell_1(B_E)\big) \otimes_{\pi_2} \big(\otimes^{n,s}_{/\pi_{n,s}\backslash} \ell_1(B_F)\big) \to \otimes^{n,s}_{/\pi_{n,s}\backslash} \big(\ell_1(B_E) \otimes_{\pi_2} \ell_1(B_F)\big),$$

given by

$$\rho\big(\big(\otimes^n u\big) \otimes \big(\otimes^n v\big)\big) = \otimes^n (u \otimes v),$$

has norm less than or equal to one.

Proof. If we take $Y = F$ and $Z = \ell_1(B_E) \otimes_{\pi_2} \ell_1(B_F)$ in Lemma 5.4, we see that the operator

$$\phi : \otimes^{n,s}_{/\pi_{n,s}\backslash} \mathcal{L}\big(\ell_1(B_F), \ell_1(B_E) \otimes_{\pi_2} \ell_1(B_F)\big) \to \mathcal{L}\big(\otimes^{n,s}_{/\pi_{n,s}\backslash} \ell_1(B_E), \otimes^{n,s}_{/\pi_{n,s}\backslash} \big(\ell_1(B_E) \otimes_{\pi_2} \ell_1(B_F)\big)\big)$$

has norm at most 1. Also the application $J : \ell_1(B_E) \to \mathcal{L}(\ell_1(B_F), \ell_1(B_E) \otimes_{\pi_2} \ell_1(B_F))$ defined by $Jz(w) = z \otimes w$ has norm one. Therefore, the norm of the map $\psi := \phi \circ \otimes^{n,s} J$ between the corresponding $/\pi_{n,s}\backslash$-tensor products is at most one.

Now, with the identification given in (10), the operator ρ is precisely B_ψ, and since (10), we conclude that ρ has norm at most one. $\square$

Now we are ready to prove that $\backslash/\pi_{n,s}\backslash/$ preserves the Banach algebra structure with Banach algebra constant equal to one. Again by Theorem 5.1, it is enough to show that, for Banach spaces E and F, the map

$$f : \big(\otimes^{n,s}_{\backslash/\pi_{n,s}\backslash/} E\big) \otimes_{\pi_2} \big(\otimes^{n,s}_{\backslash/\pi_{n,s}\backslash/} F\big) \to \otimes^{n,s}_{\backslash/\pi_{n,s}\backslash/} (E \otimes_{\pi_2} F),$$

defined by

$$f\big(\big(\otimes^n x\big) \otimes \big(\otimes^n y\big)\big) = \otimes^n (x \otimes y),$$

has norm at most one. The following diagram, where the vertical arrows are the canonical quotient maps, commutes:

$$\begin{array}{ccc} (\otimes^{n,s}_{/\pi_{n,s}\backslash} \ell_1(B_E)) \otimes_{\pi_2} (\otimes^{n,s}_{/\pi_{n,s}\backslash} \ell_1(B_F)) & \xrightarrow{\quad\rho\quad} & (\otimes^{n,s}_{/\pi_{n,s}\backslash} (\ell_1(B_E) \otimes_{\pi_2} \ell_1(B_F))) \\ \downarrow & & \downarrow \\ (\otimes^{n,s}_{\backslash/\pi_{n,s}\backslash/} E) \otimes_{\pi_2} (\otimes^{n,s}_{\backslash/\pi_{n,s}\backslash/} F) & \xrightarrow{\quad f\quad} & (\otimes^{n,s}_{\backslash/\pi_{n,s}\backslash/} (E \otimes_{\pi_2} F)) \end{array}$$

By the previous lemma, ρ has norm less than or equal to one, and so is the norm of f, since the other mappings are quotients. $\square$

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